

**CSC 151 Discrete Mathematics for Computer Science**  
**Sample Exam 2 April 30, 2018**

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name

For credit on these problems, **you must show your work** and justify your answers. Closed book, closed notes, no cell phones or computers allowed. Calculators are allowed.

1. (6 pts.) a. Consider the arithmetic sequence that begins with the five numbers 26, 37, 48, 59, 70. We are interested in expressing the sum of the first 100 terms of this sequence. Express the sum using sigma notation (don't worry about its value).

b. Find the value of the sum  $\sum_{i=1}^{50} (3i - 1)$

2. (10 pts.) Prove, using the Principle of Mathematical Induction, that  $5 \mid n^5 - n$  for all  $n \geq 1$ .

3. (6 pts.) Here are two recursive algorithms. For each, tell what the functions in general compute (not a trace of the execution) and what value will be returned by the given call.

a. 

```
def mysterya(n):  
    if n == 0:  
        return 1  
    else:  
        return n*mysterya(n-1)
```

```
print(mysterya(4))
```

b. 

```
def mysteryb(n):  
    if n == 0 or n == 1:  
        return n  
    else:  
        return mysteryb(n-1) + mysteryb(n-2)
```

```
print(mysteryb(4))
```

4. (8 pts.) Carefully prove the following:  $3n^2 + 4n + 5$  is  $\Theta(n^2)$ .

5. (2 pts. each) For which one of the growth functions,  $g(n)$ , is  $f(n) \in \Theta(g(n))$ ? For this problem choose among the following growth functions  $g(n)$ :

$n, n^2, n^3, \log n, n \log n, 1, 2^n, n!$

- a.  $f(n) = n \log n + 5n + 2^n$
- b. Let  $f(n)$  represent the number of steps to determine set membership for a number when a set is implemented using a bit string.
- c. Average case Merge sort of a list of length  $n$ .
- d. Let  $f(n)$  represent the complexity of the standard algorithm to check if a relation, represented by an  $n$  by  $n$  matrix, is reflexive.
- e. Let  $f(n)$  represent the complexity of the standard algorithm to check if a relation, represented by an  $n$  by  $n$  matrix, is transitive.
- f. Let  $f(n)$  represent how many times the statement ExecuteMe is encountered in the following code, from the algorithm for exponentiation by repeated squaring that we will see next week:

```
result = 1
x = a
while (n > 0) {
    if (n mod 2 == 1)
        result = result * x
    x = x * x
    ExecuteMe
    n =  $\lfloor n/2 \rfloor$ 
}
```

6. (12 pts.) Give a recursive definition of the following.

a. The geometric sequence: 3, 15, 75, ....

b. The set of all natural numbers divisible by 3.

c. The function  $f(n) = 5n+1$  for  $n = 0, 1, 2, \dots$

d. The set of all bit strings that have even length.

7. (3 pts.) Suppose that the number of bacteria in a colony triples every hour and starts with 1000 bacteria. Set up a recurrence relation for the number of bacteria after  $n$  hours have elapsed.

8. (4 pts.) Check one.

a. In an inductive proof of a theorem  $Q(n)$  for all  $n \geq 1$ , what must be proven in the inductive step?

\_\_\_\_\_ For all positive integers  $k$ ,  $Q(k - 1)$  implies  $Q(k)$

\_\_\_\_\_ For all positive integers  $k$ ,  $Q(k)$  implies  $Q(n)$

\_\_\_\_\_ For all positive integers  $k$ ,  $Q(k)$  implies  $Q(k + 1)$

\_\_\_\_\_ For all positive integers  $k$ ,  $Q(k)$

b. If the inductive step says that for all positive integers  $Q(k)$  implies  $Q(k+1)$ , then what is the inductive hypothesis?

\_\_\_\_\_  $Q(k+1)$

\_\_\_\_\_  $Q(n)$

\_\_\_\_\_  $Q(k)$

9. (8 pts.) Here is the matrix of a relation R. State whether it is reflexive, symmetric, anti-symmetric and/or transitive. Justify your answers.

	r	s	t	u
r	1	0	0	1
s	0	0	1	0
t	1	1	0	0
u	0	1	1	1

10. (8 pts.)

- a. Let the relation S be defined on the set of all propositions, where  $p S q$  if and only if  $p \rightarrow q$ . Determine if the relation S is reflexive, symmetric, anti-symmetric, and/or transitive. Yes or no for each with no need to justify.

\_\_\_\_\_ reflexive    \_\_\_\_\_ symmetric    \_\_\_\_\_ anti-symmetric    \_\_\_\_\_ transitive

- b. Let the relation T be defined between two sets, where  $A T B$  if and only if there exists a 1-1 onto map from A to B. Determine if the relation S is reflexive, symmetric, anti-symmetric, and/or transitive. Yes or no for each with no need to justify.

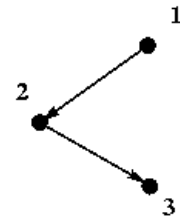
\_\_\_\_\_ reflexive    \_\_\_\_\_ symmetric    \_\_\_\_\_ anti-symmetric    \_\_\_\_\_ transitive

10 (5 pts.) Draw a digraph with 6 nodes, a, b, c, d, e, and f, that satisfy the following conditions:

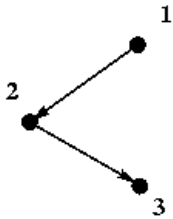
- It represents an equivalence relation.
- The equivalence relation generates 3 equivalence classes:  $[a] = [b] = [c]$ ,  $[d] = [e]$ , and  $[f]$ .

11. (6 pts.)

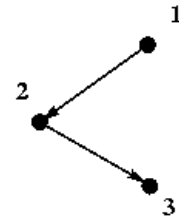
a. Draw the reflexive closure of this relation:



b. Draw the symmetric closure of this relation (below):



c. Draw the transitive closure of this relation:



12. (3 pts. each) Are the following relations equivalence relations? Justify your answers.

a. The domain of  $R$  is a group of employees at a company.  $xRy$  if  $x$  earns at least as much money as  $y$ . There are at least two employees at the company who do not earn the same amount of money.

b. The domain of relation  $R$  is the set of integers greater than 1.  $xRy$  if a positive integer other than 1 evenly divides both  $x$  and  $y$ .

13. (2 pts. each) True or false?

True      False    a.  $x$  is  $\mathbf{O}(x \log x)$ .

True      False    b. If  $f(x)$  is  $\mathbf{\Theta}(g(x))$  then  $f(x)$  is  $\mathbf{O}(g(x))$ .

True      False    c. The Halting Problem has  $n$ -factorial complexity.