CSC 151 Discrete Mathematics for Computer Science

Sample Exam 2 April 30, 2018

Solution

name

For credit on these problems, **you must show your work** and justify your answers. Closed book, closed notes, no cell phones or computers allowed. Calculators are allowed.

1. (6 pts.) a. Consider the arithmetic sequence that begins with the five numbers 26, 37, 48, 59, 70. We are interested in expressing the sum of the first 100 terms of this sequence. Express the sum using sigma notation (don't worry about its value).

$$\sum_{n=0}^{99} 26 + 11n \qquad a_n = 26 + 11n \quad n = 0, 1, \dots$$

b. Find the value of the sum $\sum_{i=1}^{50}(3i-1)$

$$\overbrace{2 + \dots + 149}^{50} = 25 \cdot (151)$$

little Gauß

2. (10 pts.) Prove, using the Principle of Mathematical Induction, that $5 \mid n^5 - n$ for all $n \geq 1$.

I Basis. The result holds when $n=1$ since $n^5 - n = 1^5 - 1 = 0$ and 5 divides 0.

II Assume 5 divides $k^5 - k$ for $k \geq 1$.
Want 5 divides $(k+1)^5 - (k+1)$

$$(k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - (k+1)$$
$$= k^5 - k + 5k^4 + 10k^3 + 10k^2 + 5k$$
$$= (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k)$$

5 divides the first expression by the IH and the second expression explicitly.
Thus 5 divides $(k+1)^5 - (k+1)$ as desired.

3. (6 pts.) Here are two recursive algorithms. For each, tell what the functions in general compute (not a trace of the execution) and what value will be returned by the given call.

a.
```
def mysterya(n):
  if n == 0:
    return 1
  else:
    return n*mysterya(n-1)
```

factorial

print(mysterya(4)) = 24

b.
```
def mysteryb(n):
  if n == 0 or n == 1:
    return n
  else:
    return mysteryb(n-1) + mysteryb(n-2)
```

fibonacci

0 1 2 3 4
0, 1, 1, 2, 3

print(mysteryb(4)) = 3

4. (8 pts.) Carefully prove the following: $3n^2 + 4n + 5$ is $O(n^2)$.

Pf/ Let $c = 12$ and $k = 1$

Then $3n^2 + 4n + 5 \le 3n^2 + 4n^2 + 5n^2 = 12n^2$ for $n \ge 1$.

So $3n^2 + 4n + 5$ is $O(n^2)$.

6. (12 pts.) Give a recursive definition of the following.

   a. The geometric sequence: 3, 15, 75, ….

$g_0 = 3$

$g_{n+1} = 5 \cdot g_n \quad n \geq 0$

   b. The set of all natural numbers divisible by 3.

Let $0 \in T$

If $n \in T$ so is $n+3$.

   c. The function f(n) = 5n+1 for n = 0, 1, 2, …

$f(0) = 1$

$f(n+1) = f(n) + 5 \quad n \geq 0$

   d. The set of all bit strings that have even length.

Let $\lambda \in E$

If $s \in E$ so is $0s1$, $0s0$, $1s0$, and $1s1$.

7. (3 pts.) Suppose that the number of bacteria in a colony triples every hour and starts with 1000 bacteria. Set up a recurrence relation for the number of bacteria after n hours have elapsed.

Check $B(0) = 1000$
$B(1) = 3000$
$B(2) = 9000$

So $B(0) = 1000$
$B(n+1) = 3 \cdot B(n) \quad n \geq$

8. (4 pts.) Check one.

a. In an inductive proof of a theorem Q(n) for all n ≥ 1, what must be proven in the inductive step?

________For all positive integers k, Q(k - 1) implies Q(k)

________For all positive integers k, Q(k) implies Q(n)

___✓___For all positive integers k, Q(k) implies Q(k + 1)

________For all positive integers k, Q(k)

b. If the inductive step says that for all positive integers Q(k) implies Q(k+1), then what is the inductive hypothesis?

________Q(k+1)

________Q(n)

___✓___Q(k)

5. (2 pts. each) For which one of the growth functions, g(n), is f(n) $\Theta(g(n))$?. For this problem choose among the following growth functions g(n):

$n, n^2, n^3, \log n, n \log n, 1, 2^n, n!$

a. $f(n) = n \log n + 5n + 2^n$

2^n biggest

b. Let f(n) represent the number of steps to determine set membership for a number when a set is implemented using a bit string.

1 constant

c. Average case Merge sort of a list of length n.

n log n

d. Let f(n) represent the complexity of the standard algorithm to check if a relation, represented by an n by n matrix, is reflexive.

n

e. Let f(n) represent the complexity of the standard algorithm to check if a relation, represented by an n by n matrix, is transitive.

n^3

f. Let f(n) represent how many times the statement ExecuteMe is encountered in the following code, from the algorithm for exponentiation by repeated squaring that we will see next week:

```
result = 1
x = a
while (n > 0) {
        if (n mod 2 == 1)
                result = result * x
        x = x * x
        ExecuteMe
        n = ⌊n/2⌋
        }
```

log n

9. (8 pts.) Here is the matrix of a relation R. State whether it is reflexive, symmetric, anti-symmetric and/or transitive. Justify your answers.

	r	s	t	u
r	1	0	0	1
s	0	0	1	0
t	1	1	0	0
u	0	1	1	1

not refl. missing a 1

not sym, mismatch

not anti-sym 1's across diag.

not trans. sRt & tRs but s R̸ s.

10. (8 pts.)

a. Let the relation S be defined on the set of all propositions, where p S q if and only if p → q. Determine if the relation S is reflexive, symmetric, anti-symmetric, and/or transitive. Yes or no for each with no need to justify.

no converse

hyp syll.

__T__ reflexive __F__ symmetric __T/F__ anti-symmetric __T__ transitive

b. Let the relation T be defined between two sets, where A T B if and only if there exists a 1-1 onto map from A to B. Determine if the relation S is reflexive, symmetric, anti-symmetric, and/or transitive. Yes or no for each with no need to justify. Think: same # of els.

__T__ reflexive __T__ symmetric __F__ anti-symmetric __T__ transitive

identity inverse {1}T{a} but {1}≠{a}. comp

10 (5 pts.) Draw a digraph with 6 nodes, a, b, c, d, e, and f, that satisfy the following conditions:

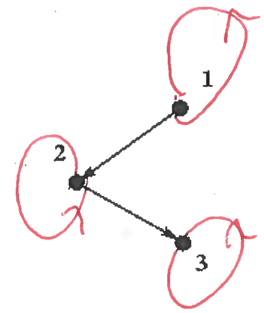
both

i. It represents an equivalence relation.

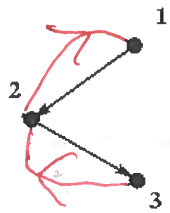
ii. The equivalence relation generates 3 equivalence classes: [a] = [b] = [c], [d] = [e], and [f].

11. (6 pts.)

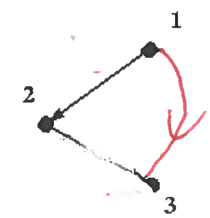
a. Draw the reflexive closure of this relation:



b. Draw the symmetric closure of this relation (below):



c. Draw the transitive closure of this relation:



12. (3 pts. each) Are the following relations equivalence relations? Justify your answers.

a. The domain of R is a group of employees at a company. xRy if x earns at least as much money as y. There are at least two employees at the company who do not earn the same amount of money.

not symmetric. If Judy makes \$50K and Ron makes \$40K, Judy R Ron but Ron $\not R$ Judy.

b. The domain of relation R is the set of integers greater than 1. xRy if a positive integer other than 1 evenly divides both x and y.

not transitive

2 divides 4 and 2 divides 6 so 4R6
3 divides 6 and 3 divides 9 so 6R9
but 4 $\not R$ 9 they have no common factors

13. (2 pts. each) True or false?

True False a. x is O(x log x). x log x is bigger than x.

True False b. If f(x) is Θ(g(x)) then f(x) is O(g(x)). $\Theta \rightarrow$ both O & Ω.

True False c. The Halting Problem has n-factorial complexity.

no complexity it is unsolvable.