# CSC 8-151 Discrete Mathematics for Computer Science
Exam 2 May 2, 2018

solution
name

For credit on these problems, **you must show your work** and justify your answers. Closed book, closed notes, no cell phones or computers allowed. Calculators are allowed.

1. (6 pts.) a. Consider the arithmetic sequence that begins with the three numbers 234, 345, 456. We are interested in expressing the sum of the first 25 terms of this sequence. Express the sum using sigma notation (don't worry about its value).

$$\sum_{n=0}^{24} 234 + 111n$$

b. Find the value of the sum $\sum_{i=1}^{100}(2i-5)$

$$50(-3+195) = 9600$$

2. (8 pts.) Prove, using the Principle of Mathematical Induction, the following formula for the sum of this geometric series:
$1 + 3 + 9 + \ldots + 3^{n-1} = ½ (3^n - 1)$ for all $n \geq 1$.

I Basis. The result holds when $n=1$ since $1 = \frac{1}{2}(3^1 - 1)$

II Assume $1+3+ + 3^{k-1} = \frac{1}{2}(3^k - 1)$

Want $1+3+\ldots+3^{k-1}+3^k = \frac{1}{2}(3^{k+1}-1)$

$1+3+\ldots+3^{k-1}+3^k = \frac{1}{2}(3^k-1)+3^k$ by the IH

$= \dfrac{3^k - 1 + 2\cdot 3^k}{2}$

$= \dfrac{3\cdot 3^k - 1}{2} = \frac{1}{2}(3^{k+1}-1)$ as desired.

3. (8 pts.) Prove, using the Principle of Mathematical Induction, that 3 evenly divides $2^{2n}-1$ for all $n \geq 1$.

Example 8.5.1

4. (6 pts) Activity 6.6.9

The matrix A given below is the adjacency matrix for a directed graph G with 4 vertices. The matrices $A^2 = A \cdot A$ and $A^3 = A \cdot A \cdot A$ have been computed using matrix multiplication. The rows and columns of the matrices are numbered 1 through 4 corresponding to the vertices in G, $G^2$, and $G^3$.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

a. Is there a walk/path of length 3 in G from vertex 1 to vertex 3? Justify your answer.

no

b. Starting from vertex 2, how many different vertices can be reached by a walk/path of length 2 in G? Justify your answer.

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5. (6 pts.) Here are two recursive algorithms. For each, tell what the functions in general compute (not a trace of the execution) and what value will be returned by the given call. Assume a and b are natural numbers.

a.
```
def mysterya(a, b):
  if b == 0:
    return 1
  else:
    return a*mysterya(a,b-1)
```

$a^b$

print(mysterya(2,3)) $= 8$

b.
```
def mysteryb(a,b):
  if b == 0 or a == b:
    return 1
  else:
    return mysteryb(a-1,b-1) + mysteryb(a-1,b)
```

Pascal's. triangle $C(n,k)$.

print(mysteryb(2,1)) $= 2$

6. (6 pts.) **Solve** these recurrence relations (express in closed form):
   a. If you invest $100 in an account bearing 2% interest compounded annually, the amount in the account after n years is modeled by the recurrence relation
      a(0) = 100
      a(n+1) = a(n)*(1.02) for $n \geq 0$

      $a(n) = 100 \cdot (1.02)^n$

   b. b(1) = 1
      b(n) = b(n-1) + n for $n \geq 2$

      $b(n) = \frac{n(n+1)}{2}$

7. (6 pts.) Carefully prove the following: $4n^2 + 7$ is $O(n^2)$.

Let $c = 11$ and $k = 1$.
Then $4n^2 + 7 \leq 4n^2 + 7n^2 = 11 \cdot n^2$ for $n \geq 1$.

8. (2 pts. each) For which one of the growth functions, g(n), is f(n) $\Theta(g(n))$ ?. For this problem choose among the following growth functions g(n):

$n, n^2, n^3, \log n, n \log n, 1, 2^n, n!$

a. f(n) = n log n + 9n

n logn

b. Let f(n) represent the number of steps to determine set membership for a number when a set is implemented using a bit string.

1

c. Average case Selection sort of a list of length n.

$n^2$

d. Let f(n) represent the complexity of the standard algorithm to check if a relation, represented by an n by n matrix, is symmetric.

$n^2$

e. Let f(n) represent the worst case complexity when using binary search on a list of length n.

log n

9. (9 pts.) Give a recursive definition of the following.
   a. The geometric sequence: 2, 14, 98, ….

$g_0 = 2$

$g_n = 7 \cdot g_{n-1} \quad n \geq 1$

   b. The function f(n) = n! for n = 0, 1, 2, …

$f(0) = 1$

$f(n) = n \cdot f(n-1) \quad n \geq 1$

   c. The **set** of all bit strings that have no zeroes.

Let $\lambda \in S$

if $s \in S$, $1s \in S$.

10. (8 pts.) Here is the matrix of a relation R. State whether it is reflexive, symmetric, anti-symmetric and/or transitive. Justify your answers.

| | r | s | t | u |
|---|---|---|---|---|
| r | 1 | 1 | 0 | 0 |
| s | 1 | 1 | 0 | 0 |
| t | 0 | 0 | 1 | 1 |
| u | 0 | 0 | 1 | 1 |

reflexive all 1's on diag
sym. match across diag
trans. $M^2 = M$ no new 1's.
not anti-s sRr rRs $r \neq s$

11. (7 pts.)

a. Let the relation S be defined on the set of all propositions, where p S q if and only if p → q. Determine if the relation S is reflexive, symmetric, anti-symmetric, and/or transitive. Yes or no for each with no need to justify.

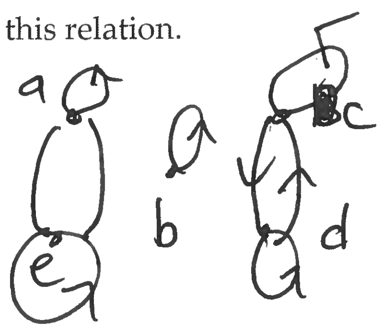
__T__ reflexive __F__ symmetric __T__ transitive

b. Let the relation T be defined between two sets, where A T B if and only if there exists a 1-1 onto map from A to B. Determine if the relation S is reflexive, symmetric, anti-symmetric, and/or transitive. Yes or no for each with no need to justify.

__T__ reflexive __T__ symmetric __F__ anti-symmetric __T__ transitive

12. (6 pts.) Let A = { a, b, c, d, e }. Consider the equivalence relation R on A:
{ (c, d), (a, e), (a, a), (b, b), (d, c), (c, c), (e, a), (d, d), (e, e) }.

a. Give the digraph representation of this relation.

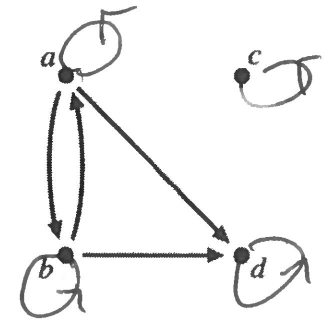



b. What is the partition of A defined by the equivalence relation R?

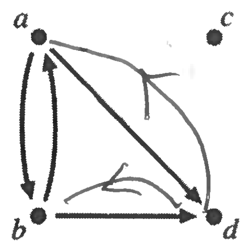
$A = \{a, e\} \cup \{b\} \cup \{c, d\}$

13. (6 pts.)

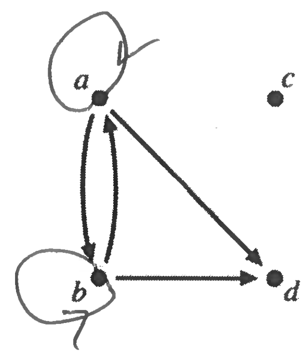
a. Draw the reflexive closure of this relation:




b. Draw the symmetric closure of this relation (below):




c. Draw the transitive closure of this relation:




14. (2 pts.) Why are there dominoes in our classroom?

Illustrates math induction

15. (2 pts. each) True or false? Circle your answer.

True False a. The function $f(n) = n$ is $\Omega(n \log n)$.

True False b. If $f(n)$ is $\Omega(g(n))$ then $g(n)$ is $O(f(n))$.

True False c. $\log_2(n)$ is $\Theta(\log_{10}(n)$