A relation R is called **asymmetric** if $(a, b) \in R$ implies that $(b, a) \notin R$. Exercises 18–24 explore the notion of an asymmetric relation. Exercise 22 focuses on the difference between asymmetry and antisymmetry.

18. Which relations in Exercise 3 are asymmetric?

19. Which relations in Exercise 4 are asymmetric?

20. Which relations in Exercise 5 are asymmetric?

21. Which relations in Exercise 6 are asymmetric?

22. Must an asymmetric relation also be antisymmetric? Must an antisymmetric relation be asymmetric? Give reasons for your answers.

23. Use quantifiers to express what it means for a relation to be asymmetric.

24. Give an example of an asymmetric relation on the set of all people.

25. How many different relations are there from a set with m elements to a set with n elements?

Let R be a relation from a set A to a set B. The **inverse relation** from B to A, denoted by R^{-1}, is the set of ordered pairs $\{(b, a) \mid (a, b) \in R\}$. The **complementary relation** $\overline{R}$ is the set of ordered pairs $\{(a, b) \mid (a, b) \notin R\}$.

26. Let R be the relation $R = \{(a, b) \mid a < b\}$ on the set of integers. Find
a) R^{-1}. b) $\overline{R}$.

27. Let R be the relation $R = \{(a, b) \mid a \text{ divides } b\}$ on the set of positive integers. Find
a) R^{-1}. b) $\overline{R}$.

28. Let R be the relation on the set of all states in the United States consisting of pairs (a, b) where state a borders state b. Find
a) R^{-1}. b) $\overline{R}$.

29. Suppose that the function f from A to B is a one-to-one correspondence. Let R be the relation that equals the graph of f. That is, $R = \{(a, f(a)) \mid a \in A\}$. What is the inverse relation R^{-1}?

30. Let $R_1 = \{(1, 2), (2, 3), (3, 4)\}$ and $R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4)\}$ be relations from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$. Find
a) $R_1 \cup R_2$. b) $R_1 \cap R_2$.
c) $R_1 - R_2$. d) $R_2 - R_1$.

31. Let A be the set of students at your school and B the set of books in the school library. Let R_1 and R_2 be the relations consisting of all ordered pairs (a, b), where student a is required to read book b in a course, and where student a has read book b, respectively. Describe the ordered pairs in each of these relations.
a) $R_1 \cup R_2$ b) $R_1 \cap R_2$
c) $R_1 \oplus R_2$ d) $R_1 - R_2$
e) $R_2 - R_1$

32. Let R be the relation $\{(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)\}$, and let S be the relation $\{(2, 1), (3, 1), (3, 2), (4, 2)\}$. Find $S \circ R$.

33. Let R be the relation on the set of people consisting of pairs (a, b), where a is a parent of b. Let S be the relation on the set of people consisting of pairs (a, b), where a and b are siblings (brothers or sisters). What are $S \circ R$ and $R \circ S$?

Exercises 34–37 deal with these relations on the set of real numbers:

$R_1 = \{(a, b) \in \mathbf{R}^2 \mid a > b\}$, the "greater than" relation,
$R_2 = \{(a, b) \in \mathbf{R}^2 \mid a \geq b\}$, the "greater than or equal to" relation,
$R_3 = \{(a, b) \in \mathbf{R}^2 \mid a < b\}$, the "less than" relation,
$R_4 = \{(a, b) \in \mathbf{R}^2 \mid a \leq b\}$, the "less than or equal to" relation,
$R_5 = \{(a, b) \in \mathbf{R}^2 \mid a = b\}$, the "equal to" relation,
$R_6 = \{(a, b) \in \mathbf{R}^2 \mid a \neq b\}$, the "unequal to" relation.

34. Find
a) $R_1 \cup R_3$. b) $R_1 \cup R_5$.
c) $R_2 \cap R_4$. d) $R_3 \cap R_5$.
e) $R_1 - R_2$. f) $R_2 - R_1$.
g) $R_1 \oplus R_3$. h) $R_2 \oplus R_4$.

35. Find
a) $R_2 \cup R_4$. b) $R_3 \cup R_6$.
c) $R_3 \cap R_6$. d) $R_4 \cap R_6$.
e) $R_3 - R_6$. f) $R_6 - R_3$.
g) $R_2 \oplus R_6$. h) $R_3 \oplus R_5$.

36. Find
a) $R_1 \circ R_1$. b) $R_1 \circ R_2$.
c) $R_1 \circ R_3$. d) $R_1 \circ R_4$.
e) $R_1 \circ R_5$. f) $R_1 \circ R_6$.
g) $R_2 \circ R_3$. h) $R_3 \circ R_3$.

37. Find
a) $R_2 \circ R_1$. b) $R_2 \circ R_2$.
c) $R_3 \circ R_5$. d) $R_4 \circ R_1$.
e) $R_5 \circ R_3$. f) $R_3 \circ R_6$.
g) $R_4 \circ R_6$. h) $R_6 \circ R_6$.

38. Let R be the parent relation on the set of all people (see Example 21). When is an ordered pair in the relation R^3?

39. Let R be the relation on the set of people with doctorates such that $(a, b) \in R$ if and only if a was the thesis advisor of b. When is an ordered pair (a, b) in R^2? When is an ordered pair (a, b) in R^n, when n is a positive integer? (Assume that every person with a doctorate has a thesis advisor.)

40. Let R_1 and R_2 be the "divides" and "is a multiple of" relations on the set of all positive integers, respectively. That is, $R_1 = \{(a, b) \mid a \text{ divides } b\}$ and $R_2 = \{(a, b) \mid a \text{ is a multiple of } b\}$. Find
a) $R_1 \cup R_2$. b) $R_1 \cap R_2$.
c) $R_1 - R_2$. d) $R_2 - R_1$.
e) $R_1 \oplus R_2$.

41. Let R_1 and R_2 be the "congruent modulo 3" and the "congruent modulo 4" relations, respectively, on the set of integers. That is, $R_1 = \{(a, b) \mid a \equiv b \pmod 3\}$ and $R_2 = \{(a, b) \mid a \equiv b \pmod 4\}$. Find
a) $R_1 \cup R_2$. b) $R_1 \cap R_2$.
c) $R_1 - R_2$. d) $R_2 - R_1$.
e) $R_1 \oplus R_2$.

42. List the 16 different relations on the set $\{0, 1\}$.

43. How many of the 16 different relations on $\{0, 1\}$ contain the pair $(0, 1)$?

44. Which of the 16 relations on $\{0, 1\}$, which you listed in Exercise 42, are
a) reflexive? b) irreflexive?
c) symmetric? d) antisymmetric?
e) asymmetric? f) transitive?

45. a) How many relations are there on the set $\{a, b, c, d\}$?
b) How many relations are there on the set $\{a, b, c, d\}$ that contain the pair (a, a)?

46. Let S be a set with n elements and let a and b be distinct elements of S. How many relations R are there on S such that
a) $(a, b) \in R$? b) $(a, b) \notin R$?
c) no ordered pair in R has a as its first element?
d) at least one ordered pair in R has a as its first element?
e) no ordered pair in R has a as its first element or b as its second element?
f) at least one ordered pair in R either has a as its first element or has b as its second element?

***47.** How many relations are there on a set with n elements that are
a) symmetric? b) antisymmetric?
c) asymmetric? d) irreflexive?
e) reflexive and symmetric?
f) neither reflexive nor irreflexive?

***48.** How many transitive relations are there on a set with n elements if
a) $n = 1$? b) $n = 2$? c) $n = 3$?

49. Find the error in the "proof" of the following "theorem."

"*Theorem*": Let R be a relation on a set A that is symmetric and transitive. Then R is reflexive.

"*Proof*": Let $a \in A$. Take an element $b \in A$ such that $(a, b) \in R$. Because R is symmetric, we also have $(b, a) \in R$. Now using the transitive property, we can conclude that $(a, a) \in R$ because $(a, b) \in R$ and $(b, a) \in R$.

50. Suppose that R and S are reflexive relations on a set A. Prove or disprove each of these statements.
a) $R \cup S$ is reflexive.
b) $R \cap S$ is reflexive.
c) $R \oplus S$ is irreflexive.
d) $R - S$ is irreflexive.
e) $S \circ R$ is reflexive.

51. Show that the relation R on a set A is symmetric if and only if $R = R^{-1}$, where R^{-1} is the inverse relation.

52. Show that the relation R on a set A is antisymmetric if and only if $R \cap R^{-1}$ is a subset of the diagonal relation $\Delta = \{(a, a) \mid a \in A\}$.

53. Show that the relation R on a set A is reflexive if and only if the inverse relation R^{-1} is reflexive.

54. Show that the relation R on a set A is reflexive if and only if the complementary relation $\overline{R}$ is irreflexive.

55. Let R be a relation that is reflexive and transitive. Prove that $R^n = R$ for all positive integers n.

56. Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs $(1, 1), (1, 2), (1, 3), (2, 3), (2, 4), (3, 1), (3, 4), (3, 5), (4, 2), (4, 5), (5, 1), (5, 2)$, and $(5, 4)$. Find
a) R^2. b) R^3. c) R^4. d) R^5.

57. Let R be a reflexive relation on a set A. Show that R^n is reflexive for all positive integers n.

***58.** Let R be a symmetric relation. Show that R^n is symmetric for all positive integers n.

59. Suppose that the relation R is irreflexive. Is R^2 necessarily irreflexive? Give a reason for your answer.

9.2 *n*-ary Relations and Their Applications

Introduction

Relationships among elements of more than two sets often arise. For instance, there is a relationship involving the name of a student, the student's major, and the student's grade point average. Similarly, there is a relationship involving the airline, flight number, starting point, destination, departure time, and arrival time of a flight. An example of such a relationship in mathematics involves three integers, where the first integer is larger than the second integer, which is larger than the third. Another example is the betweenness relationship involving points on a line, such that three points are related when the second point is between the first and the third.

We will study relationships among elements from more than two sets in this section. These relationships are called ***n*-ary relations**. These relations are used to represent computer databases. These representations help us answer queries about the information stored in databases, such as: Which flights land at O'Hare Airport between 3 A.M. and 4 A.M.? Which students at your school are sophomores majoring in mathematics or computer science and have greater than a 3.0 average? Which employees of a company have worked for the company less than 5 years and make more than $50,000?

n-ary Relations

We begin with the basic definition on which the theory of relational databases rests.

DEFINITION 1 Let $A_1, A_2, \ldots, A_n$ be sets. An *n-ary relation* on these sets is a subset of $A_1 \times A_2 \times \cdots \times A_n$. The sets $A_1, A_2, \ldots, A_n$ are called the *domains* of the relation, and n is called its *degree*.

EXAMPLE 1 Let R be the relation on $\mathbf{N} \times \mathbf{N} \times \mathbf{N}$ consisting of triples (a, b, c), where a, b, and c are integers with $a < b < c$. Then $(1, 2, 3) \in R$, but $(2, 4, 3) \notin R$. The degree of this relation is 3. Its domains are all equal to the set of natural numbers. ◀

EXAMPLE 2 Let R be the relation on $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$ consisting of all triples of integers (a, b, c) in which a, b, and c form an arithmetic progression. That is, $(a, b, c) \in R$ if and only if there is an integer k such that $b = a + k$ and $c = a + 2k$, or equivalently, such that $b - a = k$ and $c - b = k$. Note that $(1, 3, 5) \in R$ because $3 = 1 + 2$ and $5 = 1 + 2 \cdot 2$, but $(2, 5, 9) \notin R$ because $5 - 2 = 3$ while $9 - 5 = 4$. This relation has degree 3 and its domains are all equal to the set of integers. ◀

EXAMPLE 3 Let R be the relation on $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}^+$ consisting of triples (a, b, m), where a, b, and m are integers with $m \geq 1$ and $a \equiv b \pmod{m}$. Then $(8, 2, 3)$, $(-1, 9, 5)$, and $(14, 0, 7)$ all belong to R, but $(7, 2, 3)$, $(-2, -8, 5)$, and $(11, 0, 6)$ do not belong to R because $8 \equiv 2 \pmod{3}$, $-1 \equiv 9 \pmod{5}$, and $14 \equiv 0 \pmod{7}$, but $7 \not\equiv 2 \pmod{3}$, $-2 \not\equiv -8 \pmod{5}$, and $11 \not\equiv 0 \pmod{6}$. This relation has degree 3 and its first two domains are the set of all integers and its third domain is the set of positive integers. ◀

EXAMPLE 4 Let R be the relation consisting of 5-tuples (A, N, S, D, T) representing airplane flights, where A is the airline, N is the flight number, S is the starting point, D is the destination, and T is the departure time. For instance, if Nadir Express Airlines has flight 963 from Newark to Bangor at 15:00, then (Nadir, 963, Newark, Bangor, 15:00) belongs to R. The degree of this relation is 5, and its domains are the set of all airlines, the set of flight numbers, the set of cities, the set of cities (again), and the set of times. ◀

Databases and Relations

The time required to manipulate information in a database depends on how this information is stored. The operations of adding and deleting records, updating records, searching for records, and combining records from overlapping databases are performed millions of times each day in a large database. Because of the importance of these operations, various methods for representing databases have been developed. We will discuss one of these methods, called the **relational data model**, based on the concept of a relation.

A database consists of **records**, which are n-tuples, made up of **fields**. The fields are the entries of the n-tuples. For instance, a database of student records may be made up of fields containing the name, student number, major, and grade point average of the student. The relational data model represents a database of records as an n-ary relation. Thus, student records

TABLE 1 Students.

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

are represented as 4-tuples of the form (*Student_name, ID_number, Major, GPA*). A sample database of six such records is

(Ackermann, 231455, Computer Science, 3.88)
(Adams, 888323, Physics, 3.45)
(Chou, 102147, Computer Science, 3.49)
(Goodfriend, 453876, Mathematics, 3.45)
(Rao, 678543, Mathematics, 3.90)
(Stevens, 786576, Psychology, 2.99).

Relations used to represent databases are also called **tables**, because these relations are often displayed as tables. Each column of the table corresponds to an *attribute* of the database. For instance, the same database of students is displayed in Table 1. The attributes of this database are Student Name, ID Number, Major, and GPA.

A domain of an n-ary relation is called a **primary key** when the value of the n-tuple from this domain determines the n-tuple. That is, a domain is a primary key when no two n-tuples in the relation have the same value from this domain.

Records are often added to or deleted from databases. Because of this, the property that a domain is a primary key is time-dependent. Consequently, a primary key should be chosen that remains one whenever the database is changed. The current collection of n-tuples in a relation is called the **extension** of the relation. The more permanent part of a database, including the name and attributes of the database, is called its **intension**. When selecting a primary key, the goal should be to select a key that can serve as a primary key for all possible extensions of the database. To do this, it is necessary to examine the intension of the database to understand the set of possible n-tuples that can occur in an extension.

EXAMPLE 5 Which domains are primary keys for the n-ary relation displayed in Table 1, assuming that no n-tuples will be added in the future?

Solution: Because there is only one 4-tuple in this table for each student name, the domain of student names is a primary key. Similarly, the ID numbers in this table are unique, so the domain of ID numbers is also a primary key. However, the domain of major fields of study is not a primary key, because more than one 4-tuple contains the same major field of study. The domain of grade point averages is also not a primary key, because there are two 4-tuples containing the same GPA. ◀

Combinations of domains can also uniquely identify n-tuples in an n-ary relation. When the values of a set of domains determine an n-tuple in a relation, the Cartesian product of these domains is called a **composite key**.

EXAMPLE 6 Is the Cartesian product of the domain of major fields of study and the domain of GPAs a composite key for the n-ary relation from Table 1, assuming that no n-tuples are ever added?

Solution: Because no two 4-tuples from this table have both the same major and the same GPA, this Cartesian product is a composite key. ◀

Because primary and composite keys are used to identify records uniquely in a database, it is important that keys remain valid when new records are added to the database. Hence, checks should be made to ensure that every new record has values that are different in the appropriate field, or fields, from all other records in this table. For instance, it makes sense to use the student identification number as a key for student records if no two students ever have the same student identification number. A university should not use the name field as a key, because two students may have the same name (such as John Smith).

Operations on n-ary Relations

There are a variety of operations on n-ary relations that can be used to form new n-ary relations. Applied together, these operations can answer queries on databases that ask for all n-tuples that satisfy certain conditions.

The most basic operation on an n-ary relation is determining all n-tuples in the n-ary relation that satisfy certain conditions. For example, we may want to find all the records of all computer science majors in a database of student records. We may want to find all students who have a grade point average above 3.5. We may want to find the records of all computer science majors who have a grade point average above 3.5. To perform such tasks we use the selection operator.

DEFINITION 2 Let R be an n-ary relation and C a condition that elements in R may satisfy. Then the *selection operator* s_C maps the n-ary relation R to the n-ary relation of all n-tuples from R that satisfy the condition C.

EXAMPLE 7 To find the records of computer science majors in the n-ary relation R shown in Table 1, we use the operator s_{C_1}, where C_1 is the condition Major = "Computer Science." The result is the two 4-tuples (Ackermann, 231455, Computer Science, 3.88) and (Chou, 102147, Computer Science, 3.49). Similarly, to find the records of students who have a grade point average above 3.5 in this database, we use the operator s_{C_2}, where C_2 is the condition GPA > 3.5. The result is the two 4-tuples (Ackermann, 231455, Computer Science, 3.88) and (Rao, 678543, Mathematics, 3.90). Finally, to find the records of computer science majors who have a GPA above 3.5, we use the operator s_{C_3}, where C_3 is the condition (Major = "Computer Science" $\wedge$ GPA > 3.5). The result consists of the single 4-tuple (Ackermann, 231455, Computer Science, 3.88). ◀

Projections are used to form new n-ary relations by deleting the same fields in every record of the relation.

DEFINITION 3 The *projection* $P_{i_1 i_2,\ldots,i_m}$ where $i_1 < i_2 < \cdots < i_m$, maps the n-tuple $(a_1, a_2, \ldots, a_n)$ to the m-tuple $(a_{i_1}, a_{i_2}, \ldots, a_{i_m})$, where $m \le n$.

In other words, the projection $P_{i_1,i_2,\ldots,i_m}$ deletes $n - m$ of the components of an n-tuple, leaving the i_1th, i_2th, ..., and i_mth components.

TABLE 2 GPAs.

Student_name	*GPA*
Ackermann	3.88
Adams	3.45
Chou	3.49
Goodfriend	3.45
Rao	3.90
Stevens	2.99

TABLE 3 Enrollments.

Student	*Major*	*Course*
Glauser	Biology	BI 290
Glauser	Biology	MS 475
Glauser	Biology	PY 410
Marcus	Mathematics	MS 511
Marcus	Mathematics	MS 603
Marcus	Mathematics	CS 322
Miller	Computer Science	MS 575
Miller	Computer Science	CS 455

TABLE 4 Majors.

Student	*Major*
Glauser	Biology
Marcus	Mathematics
Miller	Computer Science

EXAMPLE 8 What results when the projection $P_{1,3}$ is applied to the 4-tuples $(2, 3, 0, 4)$, (Jane Doe, 234111001, Geography, 3.14), and (a_1, a_2, a_3, a_4)?

Solution: The projection $P_{1,3}$ sends these 4-tuples to $(2, 0)$, (Jane Doe, Geography), and (a_1, a_3), respectively. ◀

Example 9 illustrates how new relations are produced using projections.

EXAMPLE 9 What relation results when the projection $P_{1,4}$ is applied to the relation in Table 1?

Solution: When the projection $P_{1,4}$ is used, the second and third columns of the table are deleted, and pairs representing student names and grade point averages are obtained. Table 2 displays the results of this projection. ◀

Fewer rows may result when a projection is applied to the table for a relation. This happens when some of the n-tuples in the relation have identical values in each of the m components of the projection, and only disagree in components deleted by the projection. For instance, consider the following example.

EXAMPLE 10 What is the table obtained when the projection $P_{1,2}$ is applied to the relation in Table 3?

Solution: Table 4 displays the relation obtained when $P_{1,2}$ is applied to Table 3. Note that there are fewer rows after this projection is applied. ◀

The **join** operation is used to combine two tables into one when these tables share some identical fields. For instance, a table containing fields for airline, flight number, and gate, and another table containing fields for flight number, gate, and departure time can be combined into a table containing fields for airline, flight number, gate, and departure time.

DEFINITION 4 Let R be a relation of degree m and S a relation of degree n. The *join* $J_p(R, S)$, where $p \le m$ and $p \le n$, is a relation of degree $m + n - p$ that consists of all $(m + n - p)$-tuples $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$, where the m-tuple $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p)$ belongs to R and the n-tuple $(c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$ belongs to S.

In other words, the join operator J_p produces a new relation from two relations by combining all m-tuples of the first relation with all n-tuples of the second relation, where the last p components of the m-tuples agree with the first p components of the n-tuples.

TABLE 5 Teaching_assignments.

Professor	Department	Course_number
Cruz	Zoology	335
Cruz	Zoology	412
Farber	Psychology	501
Farber	Psychology	617
Grammer	Physics	544
Grammer	Physics	551
Rosen	Computer Science	518
Rosen	Mathematics	575

TABLE 6 Class_schedule.

Department	Course_number	Room	Time
Computer Science	518	N521	2:00 P.M.
Mathematics	575	N502	3:00 P.M.
Mathematics	611	N521	4:00 P.M.
Physics	544	B505	4:00 P.M.
Psychology	501	A100	3:00 P.M.
Psychology	617	A110	11:00 A.M.
Zoology	335	A100	9:00 A.M.
Zoology	412	A100	8:00 A.M.

EXAMPLE 11 What relation results when the join operator J_2 is used to combine the relation displayed in Tables 5 and 6?

Solution: The join J_2 produces the relation shown in Table 7. ◀

There are other operators besides projections and joins that produce new relations from existing relations. A description of these operations can be found in books on database theory.

SQL

Links

The database query language SQL (short for Structured Query Language) can be used to carry out the operations we have described in this section. Example 12 illustrates how SQL commands are related to operations on n-ary relations.

EXAMPLE 12 We will illustrate how SQL is used to express queries by showing how SQL can be employed to make a query about airline flights using Table 8. The SQL statement

```
SELECT Departure_time
FROM Flights
WHERE Destination='Detroit'
```

is used to find the projection P_5 (on the Departure_time attribute) of the selection of 5-tuples in the Flights database that satisfy the condition: Destination = 'Detroit'. The output would be a list containing the times of flights that have Detroit as their destination, namely, 08:10, 08:47, and 09:44. SQL uses the FROM clause to identify the n-ary relation the query is applied to, the WHERE clause to specify the condition of the selection operation, and the SELECT clause to specify the projection operation that is to be applied. (*Beware:* SQL uses SELECT to represent a projection, rather than a selection operation. This is an unfortunate example of conflicting terminology.) ◀

TABLE 7 Teaching_schedule.

Professor	Department	Course_number	Room	Time
Cruz	Zoology	335	A100	9:00 A.M.
Cruz	Zoology	412	A100	8:00 A.M.
Farber	Psychology	501	A100	3:00 P.M.
Farber	Psychology	617	A110	11:00 A.M.
Grammer	Physics	544	B505	4:00 P.M.
Rosen	Computer Science	518	N521	2:00 P.M.
Rosen	Mathematics	575	N502	3:00 P.M.

TABLE 8 Flights.

Airline	Flight_number	Gate	Destination	Departure_time
Nadir	122	34	Detroit	08:10
Acme	221	22	Denver	08:17
Acme	122	33	Anchorage	08:22
Acme	323	34	Honolulu	08:30
Nadir	199	13	Detroit	08:47
Acme	222	22	Denver	09:10
Nadir	322	34	Detroit	09:44

Example 13 shows how SQL queries can be made involving more than one table.

EXAMPLE 13 The SQL statement

```
SELECT Professor, Time
FROM Teaching_assignments, Class_schedule
WHERE Department='Mathematics'
```

is used to find the projection $P_{1,5}$ of the 5-tuples in the database (shown in Table 7), which is the join J_2 of the Teaching_assignments and Class_schedule databases in Tables 5 and 6, respectively, which satisfy the condition: Department = Mathematics. The output would consist of the single 2-tuple (Rosen, 3:00 P.M.). The SQL FROM clause is used here to find the join of two different databases. ◀

We have only touched on the basic concepts of relational databases in this section. More information can be found in [AhUl95].

Exercises

1. List the triples in the relation $\{(a, b, c) \mid a, b, \text{ and } c$ are integers with $0 < a < b < c < 5\}$.
2. Which 4-tuples are in the relation $\{(a, b, c, d) \mid a, b, c,$ and d are positive integers with $abcd = 6\}$?
3. List the 5-tuples in the relation in Table 8.
4. Assuming that no new n-tuples are added, find all the primary keys for the relations displayed in
   a) Table 3. b) Table 5.
   c) Table 6. d) Table 8.
5. Assuming that no new n-tuples are added, find a composite key with two fields containing the *Airline* field for the database in Table 8.
6. Assuming that no new n-tuples are added, find a composite key with two fields containing the Professor field for the database in Table 7.
7. The 3-tuples in a 3-ary relation represent the following attributes of a student database: student ID number, name, phone number.
   a) Is student ID number likely to be a primary key?
   b) Is name likely to be a primary key?
   c) Is phone number likely to be a primary key?
8. The 4-tuples in a 4-ary relation represent these attributes of published books: title, ISBN, publication date, number of pages.
   a) What is a likely primary key for this relation?
   b) Under what conditions would (title, publication date) be a composite key?
   c) Under what conditions would (title, number of pages) be a composite key?

9. The 5-tuples in a 5-ary relation represent these attributes of all people in the United States: name, Social Security number, street address, city, state.

a) Determine a primary key for this relation.

b) Under what conditions would (name, street address) be a composite key?

c) Under what conditions would (name, street address, city) be a composite key?

10. What do you obtain when you apply the selection operator s_C, where C is the condition Room $=$ A100, to the database in Table 7?

11. What do you obtain when you apply the selection operator s_C, where C is the condition Destination $=$ Detroit, to the database in Table 8?

12. What do you obtain when you apply the selection operator s_C, where C is the condition (Project $= 2$) $\wedge$ (Quantity $\geq$ 50), to the database in Table 10?

13. What do you obtain when you apply the selection operator s_C, where C is the condition (Airline $=$ Nadir) $\vee$ (Destination $=$ Denver), to the database in Table 8?

14. What do you obtain when you apply the projection $P_{2,3,5}$ to the 5-tuple (a, b, c, d, e)?

15. Which projection mapping is used to delete the first, second, and fourth components of a 6-tuple?

16. Display the table produced by applying the projection $P_{1,2,4}$ to Table 8.

17. Display the table produced by applying the projection $P_{1,4}$ to Table 8.

18. How many components are there in the n-tuples in the table obtained by applying the join operator J_3 to two tables with 5-tuples and 8-tuples, respectively?

19. Construct the table obtained by applying the join operator J_2 to the relations in Tables 9 and 10.

20. Show that if C_1 and C_2 are conditions that elements of the n-ary relation R may satisfy, then $s_{C_1 \wedge C_2}(R) = s_{C_1}(s_{C_2}(R))$.

21. Show that if C_1 and C_2 are conditions that elements of the n-ary relation R may satisfy, then $s_{C_1}(s_{C_2}(R)) = s_{C_2}(s_{C_1}(R))$.

22. Show that if C is a condition that elements of the n-ary relations R and S may satisfy, then $s_C(R \cup S) = s_C(R) \cup s_C(S)$.

23. Show that if C is a condition that elements of the n-ary relations R and S may satisfy, then $s_C(R \cap S) = s_C(R) \cap s_C(S)$.

24. Show that if C is a condition that elements of the n-ary relations R and S may satisfy, then $s_C(R - S) = s_C(R) - s_C(S)$.

25. Show that if R and S are both n-ary relations, then $P_{i_1,i_2,\ldots,i_m}(R \cup S) = P_{i_1,i_2,\ldots,i_m}(R) \cup P_{i_1,i_2,\ldots,i_m}(S)$.

26. Give an example to show that if R and S are both n-ary relations, then $P_{i_1,i_2,\ldots,i_m}(R \cap S)$ may be different from $P_{i_1,i_2,\ldots,i_m}(R) \cap P_{i_1,i_2,\ldots,i_m}(S)$.

27. Give an example to show that if R and S are both n-ary relations, then $P_{i_1,i_2,\ldots,i_m}(R - S)$ may be different from $P_{i_1,i_2,\ldots,i_m}(R) - P_{i_1,i_2,\ldots,i_m}(S)$.

28. a) What are the operations that correspond to the query expressed using this SQL statement?

```
SELECT Supplier
FROM Part_needs
WHERE 1000 ≤ Part_number ≤ 5000
```

b) What is the output of this query given the database in Table 9 as input?

29. a) What are the operations that correspond to the query expressed using this SQL statement?

```
SELECT Supplier, Project
FROM Part_needs, Parts_inventory
WHERE Quantity ≤ 10
```

b) What is the output of this query given the databases in Tables 9 and 10 as input?

30. Determine whether there is a primary key for the relation in Example 2.

31. Determine whether there is a primary key for the relation in Example 3.

32. Show that an n-ary relation with a primary key can be thought of as the graph of a function that maps values of the primary key to $(n-1)$-tuples formed from values of the other domains.

TABLE 9 Part_needs.

Supplier	*Part_number*	*Project*
23	1092	1
23	1101	3
23	9048	4
31	4975	3
31	3477	2
32	6984	4
32	9191	2
33	1001	1

TABLE 10 Parts_inventory.

Part_number	*Project*	*Quantity*	*Color_code*
1001	1	14	8
1092	1	2	2
1101	3	1	1
3477	2	25	2
4975	3	6	2
6984	4	10	1
9048	4	12	2
9191	2	80	4

9.3 Representing Relations

Introduction

In this section, and in the remainder of this chapter, all relations we study will be binary relations. Because of this, in this section and in the rest of this chapter, the word relation will always refer to a binary relation. There are many ways to represent a relation between finite sets. As we have seen in Section 9.1, one way is to list its ordered pairs. Another way to represent a relation is to use a table, as we did in Example 3 in Section 9.1. In this section we will discuss two alternative methods for representing relations. One method uses zero–one matrices. The other method uses pictorial representations called directed graphs, which we will discuss later in this section.

Generally, matrices are appropriate for the representation of relations in computer programs. On the other hand, people often find the representation of relations using directed graphs useful for understanding the properties of these relations.

Representing Relations Using Matrices

A relation between finite sets can be represented using a zero–one matrix. Suppose that R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$. (Here the elements of the sets A and B have been listed in a particular, but arbitrary, order. Furthermore, when $A = B$ we use the same ordering for A and B.) The relation R can be represented by the matrix $\mathbf{M}_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

In other words, the zero–one matrix representing R has a 1 as its (i, j) entry when a_i is related to b_j, and a 0 in this position if a_i is not related to b_j. (Such a representation depends on the orderings used for A and B.)

The use of matrices to represent relations is illustrated in Examples 1–6.

EXAMPLE 1 Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R if $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

Solution: Because $R = \{(2, 1), (3, 1), (3, 2)\}$, the matrix for R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The 1s in $\mathbf{M}_R$ show that the pairs (2, 1), (3, 1), and (3, 2) belong to R. The 0s show that no other pairs belong to R. ◀

EXAMPLE 2 Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$