

**MAT 1-221 Linear Algebra**  
**Old Sample Exam 2**

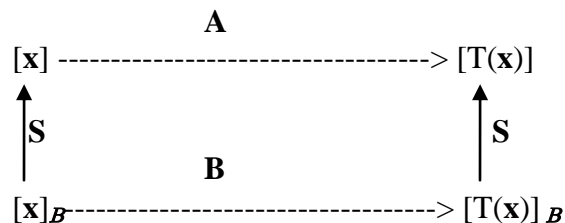
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name

Put your answers on the paper provided. Please work neatly and staple your solutions, in problem order, to these two pages. **You must show your work and justify your answers for credit.** You are free to use your calculators to compute rref, inverses, and matrix arithmetic; you may not store and use information such as formulas or definitions in your calculator. When showing your work, simply indicate where you have used a calculator.

- (3 pts. each) Definitions:
  - A basis of a linear space  $V$  is a linear independent spanning set. Complete the definition: The set of vectors  $v_1, v_2, \dots, v_n$  in  $V$  is **linearly independent** if
  - Complete the definition: The matrix  $A$  is **similar** to the matrix  $B$  if
- (10 pts) Determine a basis for the kernel and a basis for the image of the following matrix:
$$\begin{matrix} 1 & 2 & 0 & 3 & 0 \\ 2 & 4 & 1 & 9 & 5 \end{matrix}$$
- (8 pts) Prove that there is a non-trivial relation among the vectors  $v_1, v_2, \dots, v_n$  if and only if at least one of the vectors  $v_i$  is a linear combination of the other vectors  $v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n$ .
- (4 pts. each) True or false? Justify your answers.
  - The set of all invertible 3 by 3 matrices is a subspace of  $\mathbf{R}^{3 \times 3}$
  - The set of all  $f(x)$  such that  $f(1) = 0$  is a subspace of  $\mathbf{C}^\infty$ .
- (4 pts. each) True or false? Justify your answers.
  - The transformation  $S: V \rightarrow V$  given by  $S(x_0, x_1, x_2, x_3, \dots) = (x_1, x_3, x_5, \dots)$  (drop all even terms) is a linear transformation, where  $V$  is the space of infinite sequences of real numbers.
  - The transformation  $T: \mathbf{P}_2 \rightarrow \mathbf{R}$  given by  $T(p) = \int_0^1 p(t) dt$  is a linear transformation.
- (4 pts. each) The derivative operator  $Df = f'$  is a linear operator from  $\mathbf{P}_2$  to  $\mathbf{P}_2$ .
  - Describe the kernel of  $D$  and give a basis for the kernel.
  - Describe the image of  $D$  and give a basis for the image.

The following diagram characterizes change of basis and the associated matrices of a linear transformation:



Problems 7 and 8 refer to this diagram.

- 7 (10 pts.) a. Given the vector  $[2 \ 3]$  in  $\mathbf{R}^2$  (top left) and the basis  $[1, 1]$  and  $[-1,1]$  of  $\mathbf{R}^2$  find  $[\mathbf{x}]_{\mathbf{B}}$ .  
 b. With the same basis as in part a, suppose you are given  $[\mathbf{x}]_{\mathbf{B}} = [4 \ 5]_{\mathbf{B}}$ . What is the vector with respect to the standard basis? Show your work.

c. Describe the matrix  $\mathbf{S}$ . Is it always an invertible matrix? Justify your answer.

- 8 (9 pts.) Let  $T$  be the linear transformation given by multiplication by the matrix. Give the matrix,  $\mathbf{B}$ , of this transformation relative to the basis  $[1 \ 1 \ 1]$ ,  $[0 \ 1 \ 2]$  and  $[1 \ 2 \ 4]$ . Compute  $\mathbf{B}$  in two ways, by determining the effect of  $T$  on the basis elements, and using matrix multiplication with the diagram above.

0	2	-1
2	-1	0
4	-4	1

9. (3 pts. each) True or false? Briefly justify each answer.

- There exists a 3 by 3 matrix  $\mathbf{A}$  with  $\ker(\mathbf{A}) = \text{im}(\mathbf{A})$ .
- If  $2\mathbf{u} + 3\mathbf{v} + 4\mathbf{w} = 5\mathbf{u} + 6\mathbf{v} + 7\mathbf{w}$  then vectors  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  must be linearly dependent.
- If vectors  $v_1, v_2, v_3$ , and  $v_4$  are linearly independent then so are vectors  $v_1, v_2$ , and  $v_3$ .
- $\{1 - t, t - t^2, t^2 - 1\}$  is a basis for  $\mathbf{P}_2$ .
- If matrix  $\mathbf{A}$  is similar to matrix  $\mathbf{B}$ , and matrix  $\mathbf{B}$  is similar to matrix  $\mathbf{C}$ , then matrix  $\mathbf{A}$  is similar to matrix  $\mathbf{C}$ .
- If  $\mathbf{A}$  is an invertible  $n$  by  $n$  matrix then the kernels of  $\mathbf{A}$  and  $\mathbf{A}^{-1}$  must be the same.
- If  $\mathbf{A}$  is a matrix with  $m$  rows and  $n$  columns then  $\text{rank } \mathbf{A} + \text{nullity } \mathbf{A} = n$ .

10. (3 pts. each) Give an example that satisfies the following conditions. Explicitly demonstrate that your examples satisfy the given conditions. If the given conditions cannot be satisfied, say so and briefly explain why.

- An infinite dimensional linear space.
- A subspace of  $\mathbf{R}^{2 \times 3}$  of dimension 2.
- A basis of  $\mathbf{P}_3$  that includes  $1 + t + t^2$
- A basis for the, the set of all vectors in  $\mathbf{R}^3$  that are perpendicular to  $(1,2,3)$ .