Chapter 24 - Electric Potential and Potential Energy

Electric Potential Energy

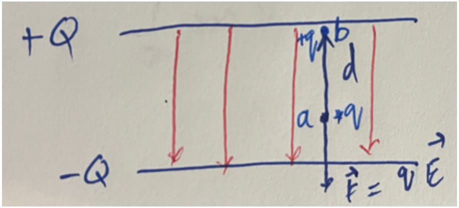
Let's first visit the gravitational potential energy. We know that when an object move in gravitational field from point a to point b then the change in the potential energy is:

$$\Delta U = -\int_a^b \vec{F} \cdot d\vec{r}$$

where $\vec{F}$ is gravitational force, and the negative sign indicate that potential energy will increase if $d\vec{r}$ is opposite to the direction of gravitational force.

Electric Potential Energy

This general expression can be applied to the change in electric potential energy inside an electric field as well, but then the force in the equation will be the electric force.



Electric Potential Energy

Let say that a charge q move from point a to point b in this uniform electric field, then the change in electric potential energy of the charge in the field according to the above equation will be

$$\Delta U = -\int_a^b q\vec{E} \cdot d\vec{r}$$

Since E is uniform it is not changing, then we can take it out of the integration. Also $\vec{E}$ and $d\vec{r}$ are in opposite direction so the above equation will become:

$$\Delta U = qE \int_a^b dr = qEd$$

So change in electric potential energy is qEd which is similar to change in gravitational potential energy mgh.

Potential Energy of Point Charges

Electric potential energy of any charge configuration is equal to the work required by external agent to bring these charges together to the current configuration from the configuration where the electric potential energy was zero.

$$\Delta U = W_{ext} = -W_e$$

Where W_e is electric force. The minus sign indicate that in order to increase the potential energy, the external force should be against the electric force.

$$\Delta U = -\int_{\infty}^{r} \vec{F} \cdot d\vec{r}$$

If there are only two charges q_1 and q_2 then

$$\Delta U = -\int_{\infty}^{r} \frac{kq_1 q_2}{r^2} dr$$

$$U = \frac{kq_1 q_2}{r}$$

Potential Energy of Point Charges

This is the electric potential energy between two point charges separated by a distance r.

If there are three charges then the total potential energy of the three charges configuration will be

$$U = \frac{kq_1 q_2}{r_{12}} + \frac{kq_1 q_3}{r_{13}} + \frac{kq_2 q_3}{r_{23}}$$

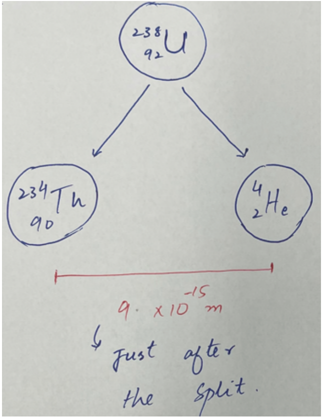
We can extend the formula for any number of charges. For many charges

$$U = \sum_{\text{all pairs}} \frac{kq_i q_j}{r_{ij}}$$

Example: Uranium Split

Uranium nucleus splits into two pieces (Thorium and Helium). Assume both nuclei are at rest right after the uranium split. Just after the split the distance between the product nuclei is 9×10^{-15} m. How fast is the Helium nucleus moving when it is 2×10^{-14} m away from the Thorium nucleus. Assume that the Thorium nucleus remains at rest. (It does not but the assumption is a good approximation)

Example: Uranium Split



Example: Uranium Split

$$U_i = \frac{(9 \times 10^9)(90e)(2e)}{9 \times 10^{-15}} = 4.6 \times 10^{-12} \text{ J}$$

$$U_b = \frac{(9 \times 10^9)(90e)(2e)}{2 \times 10^{-14}} = 2.1 \times 10^{-12} \text{ J}$$

$$\frac{1}{2} m_{He} v^2 + 2.1 \times 10^{-12} = 4.6 \times 10^{-12}$$

$$m_{He} = 4 \times 1.67 \times 10^{-27} = 6.68 \times 10^{-27} \text{ kg}$$

Example: Uranium Split

$$v^2 = \frac{2(4.6 \times 10^{-12} - 2.1 \times 10^{-12})}{6.68 \times 10^{-27}} \Rightarrow v = 2.7 \times 10^7 \text{ m/s}$$

Very fast. About 20% the speed of light.

Electrostatic Potential

Electrostatic potential is defined as PE per unit charge.

$$V = \frac{U}{q}$$

It is a scalar quantity. V is measured in volts: $\frac{J}{C}$

$$1 \text{ volt} = \frac{1J}{1C}$$

$$U = qV \quad \text{and} \quad \Delta U = q\Delta V$$

Potential from E-field

$$\Delta V = \frac{\Delta U}{q} = -\int_i^f \vec{E} \cdot d\vec{r}$$

$$\boxed{V = -\int_\infty^f \vec{E} \cdot d\vec{r}}$$

Advantage: $V(\vec{r})$ is a scalar: no components.

Voltage

Voltage: is the difference in potential between two points.

$$\text{Voltage} = \Delta V = V_B - V_A$$

Electron Volt Definition

Electron volt is a unit of energy.
If an electron passes through a potential difference(voltage) of 1 volt, then the change (gain or loss) in it's energy will be 1 eV.

$$\Delta U = q\Delta V$$

$$\Delta U = (1e)(1V) = (1.6 \times 10^{-19} C)(1V) = 1.6 \times 10^{-19} J$$

$$\boxed{1\text{eV} = 1.6 \times 10^{-19}\ \text{J}}$$

Example: Accelerated Electron

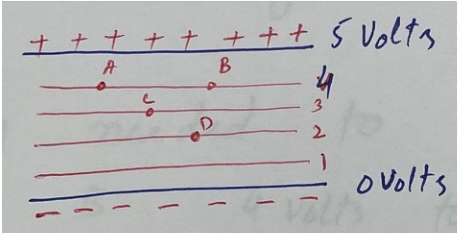
An electron is accelerated by 20,000 volts. What will be it's final speed?

$$U = (1.6 \times 10^{-19} C)(20,000 \text{ volts}) = 3.2 \times 10^{-15} J$$

$$\frac{1}{2} m_e v^2 = 3.2 \times 10^{-15} \Rightarrow v = 8.4 \times 10^7 \text{ m/s}$$

Equipotential Surfaces

These are surfaces across which the potential $V(\vec{r})$ is constant. As $\Delta V = 0$, if we move a particle along an equipotential surface, this means no energy is gained or lost, so no work is required.
In the figure, no work is required to move an electron from A to B.
Work is required to move it from B to C.



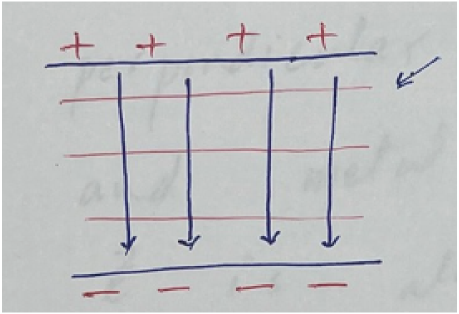
Equipotential Surfaces

Energy needed to move an electron from B (4 volts) to C (3 volts):

$$1eV = 1.6 \times 10^{-19} J$$

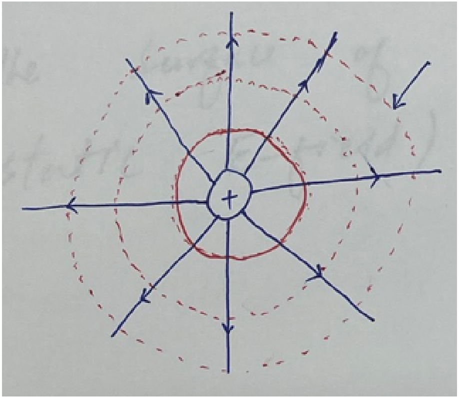
Equipotential Surfaces

Equipotentials are always perpendicular to the E-field.



Equipotential Surfaces

For a positive point charge, the electric field is perpendicular to equipotential surfaces, which are spherical closed surfaces in 3D.



Equipotential Surfaces

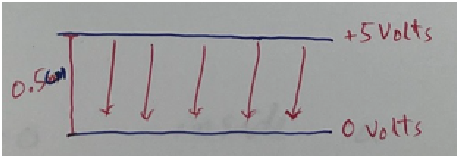
As $\vec{E} = 0$ inside a conductor:

$$\Delta V = \int \vec{E} \cdot d\vec{r} = 0, \quad \text{inside a conductor}$$

If the change in potential is zero inside a conductor, the entire conductor is at the same potential (Eqipotential).
Also, $\vec{E}$ is perpendicular to the conductor surface. If it weren't, the parallel component would move charges until it vanishes.
$=>$ Conductors are equipotential.

Example: E-field Between Plates

Example: Plates separated by $l = 0.5 \times 10^{-2} m$ charged to 5 volts.



What is electric field between the plates?

Solution: E-field Between Plates

$$V = -\int \vec{E} \cdot d\vec{r}, \quad E \text{ is constant}$$

$$V = -E(\Delta r) = -E(5 \times 10^{-3})$$

$$5 = -E(5 \times 10^{-3}) \Rightarrow E = -\frac{5}{5 \times 10^{-3}} = -10^{3}$$

$$\boxed{E = 1000 \frac{V}{m}}$$

$\frac{V}{m} = \frac{N}{C}$, Proof

$\frac{V}{m}$ is another unit of E field which is the same as $\frac{N}{C}$

$$\frac{\Delta V}{d} = \frac{q(J/C)}{meter} = \frac{N \cdot meter}{C \cdot meter} = \frac{N}{C}$$

E Field from Potential

Since derivative & integration are reciprocal of each other so this means that potential is the integration of E-field over $\vec{r}$, then E-field is the derivative of V over $\vec{r}$.

$$\boxed{\vec{E} = -\frac{dV}{d\vec{r}} \quad (\text{since } V = -\int \vec{E} \cdot d\vec{r})}$$

E Field From Potential

More formally (in three dimension)

$$E_x = -\frac{dV}{dx}, \quad E_y = -\frac{dV}{dy}, \quad E_z = -\frac{dV}{dz}$$

E Field From Potential

let's see how this works.

Suppose we have a point charge $+Q$ and a test charge $+q$. They are separated by a distance r, then the electrostatic potential energy of the configuration is

$$U = \frac{kQq}{r}$$

Electrostatic potential is

$$V = \frac{U}{q} = \frac{kQ}{r}$$

E Field From Potential

This is potential of charge Q at a distance r. Now what will be electric field due to $+Q$ at a distance r away from the charge.

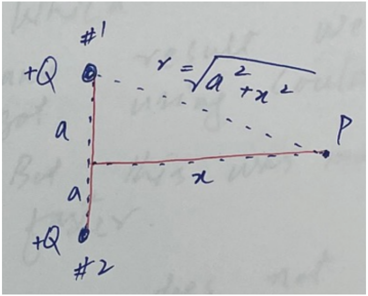
$$E = -\frac{dV}{dr} = -\frac{d}{dr}\left(\frac{kQ}{r}\right) = \frac{kQ}{r^2}$$

$$\boxed{E = \frac{kQ}{r^2}}$$

This is the correct answer because this is electric field due to a point charge.

E Field From Potential

Suppose we have two point charges, as shown in the figure below



E Field From Potential

Since V is a scalar quantity, then the electrostatic potential at point P due to these charges will be

$$V_1 = \frac{kQ}{\sqrt{a^2 + x^2}}, \qquad V_2 = \frac{kQ}{\sqrt{a^2 + x^2}},$$

$$V = V_1 + V_2 = \frac{2kQ}{\sqrt{a^2 + x^2}}$$

What is E-field at this point

$$E = -\frac{dV}{dr}$$

E Field From Potential

$$E = -2kQ\frac{d}{dx}(a^2 + x^2)^{-1/2} = -2kQ(-1/2)(a^2 + x^2)^{-3/2}(2x)$$

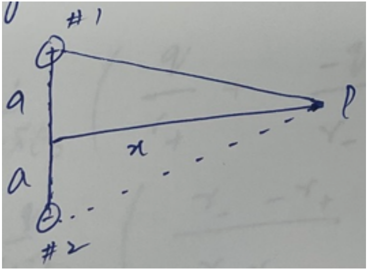
$$E = \frac{kQ(2x)}{(x^2 + a^2)^{3/2}} = \frac{2kQx}{(x^2 + a^2)^{3/2}}$$

$$\boxed{E = \frac{2kQx}{(x^2 + a^2)^{3/2}}}$$

Which is the same result we got using Coulomb's Law. But this was much faster.

E Field From Potential

This trick, unfortunately, does not always work.
For example, lets check for a dipole



E Field From Potential

$$V_1 = \frac{kQ}{r}, \quad V_2 = \frac{-kQ}{r}$$

$$V_x = 0 \Rightarrow E = \frac{-dV}{dx} \Rightarrow \boxed{E_x = 0}$$

which is true because the x-component of E-field is zero at "P" but y-comp is not. Then we need to find V as a function of x and y (which will be some other point, not the point on the x-axis) and then take derivative to find E. But in this case it is easy to use Coulomb's Law, find the x & y components and then add all the components vectorially.

Electric Potential Due to a Dipole

Suppose we have an electric dipole as shown in the figure below.

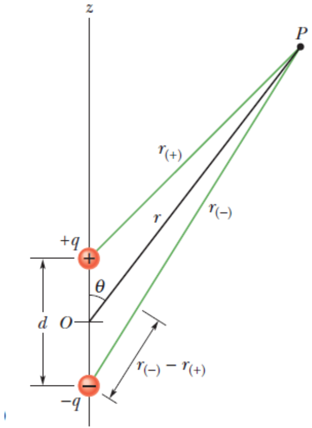


Figure 1: Figure adopted from HRW

Electric Potential Due to a Dipole

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_+} + \frac{-q}{r_-}\right), \quad V = \frac{q}{4\pi\varepsilon_0}\left(\frac{r_- - r_+}{r_+ r_-}\right)$$

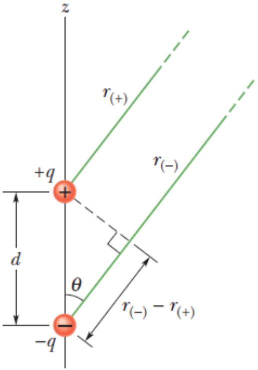


Figure 2: Figure adopted from HRW

Electric Potential Due to a Dipole

If $r \gg d$ (this means if the dipole is very small compare to the r) $=>$ $r_- - r_+ \approx d\cos\theta$, also $r_+ r_- = r^2$. These approximations can easily be seen from the figure below.

$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{d\cos\theta}{r^2}\right)$$

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}}$$

where $p = qd$, is electric dipole moment.

Potential due to a spherical shell (Continuous Charge Distribution)

Now, let's consider continuous charge distribution and find the electric potential. Suppose we have a spherical shell of radius R.

Potential due to a spherical shell

We know that

$$\vec{E}(r) = \frac{kQ}{r^2}\hat{r}, \quad for \quad r > R$$

$$\vec{E}(r) = 0, \quad for \quad r < R$$

Potential due to a spherical shell

Now, to find potential outside (for $r > R$)

$$V(r) = -\int \vec{E}(r) \cdot d\vec{r}$$

$$V(r) = -\int \vec{E}(r) d\vec{r} = -kQ \int_{\infty}^{r} \left(\frac{1}{r}\right) dr$$

$$V(r) = -kQ(-1/r) = \frac{kQ}{r}$$

$$\boxed{V(r) = \frac{kQ}{r}} \qquad \text{For } r > R$$

Potential due to a spherical shell (Continuous Charge Distribution

Now inside: For $r < R$

$$V(r) = -\int_{\infty}^{r} \vec{E}(r) d\vec{r} = -\left[\int_{\infty}^{R} \vec{E}(r) d\vec{r} + \int_{R}^{r} \vec{E}(r) d\vec{r}\right]$$

$$V(r) = -\int_{\infty}^{R} \vec{E}(r) d\vec{r} + 0, \qquad (since \vec{E}(r) = 0 inside)$$

$$V(r) = -\int_{\infty}^{R} \frac{kQ}{r^2} dr$$

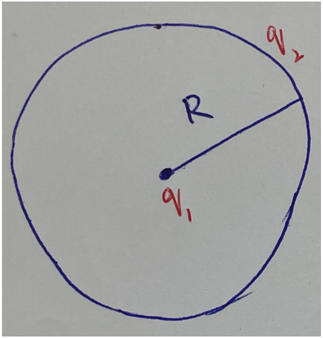
Potential due to a spherical shell (Continuous Charge Distribution

$$V(r) = kQ\left(\frac{1}{r} - \frac{1}{R}\right) + \frac{kQ}{R}$$

$$\boxed{V(r) = \frac{kQ}{R}} \qquad \text{For } r < R$$

A Hollow shell with a point charge at the cente (Continuous Charge Distribution.

Let suppose that now there is a point charge q_1 at the center of the spherical shell as shown in the figure below



A Hollow shell with a point charge at the center (Continuous Charge Distribution)

$$\vec{E} = \frac{k(q_1 + q_2)}{r^2}, \quad for \quad r > R$$

$$\vec{E} = \frac{kq_1}{r^2}, \quad for \quad r < R$$

$V(r) =?$

A Hollow shell with a point charge at the center (Continuous Charge Distribution)

For $r > R$

$$V(r) = -\int_{\infty}^{r} \vec{E}(r) d\vec{r} = \int_{\infty}^{r} \frac{k(q_1 + q_2)}{r^2} dr$$

$$V(r) = \frac{k(q_1 + q_2)}{r}$$

A Hollow shell with a point charge at the center (Continuous Charge Distribution)

For $r < R$

$$V(r) = -\int_{\infty}^{r} \vec{E}(r)d\vec{r} = -\int_{\infty}^{R} \vec{E}(r)d\vec{r} - \int_{R}^{r} \vec{E}(r)d\vec{r}$$

$$V(r) = -\int_{\infty}^{R} \frac{k(q_1+q_2)}{r^2}dr - \int_{R}^{r} \frac{kq_1}{r^2}$$

$$V(r) = \frac{k(q_1+q_2)}{R} + kq_1\left(\frac{1}{r} - \frac{1}{R}\right)$$

A Hollow shell with a point charge at the center (Continuous Charge Distribution)

$$\boxed{V(r) = \frac{kq_1}{r} + \frac{kq_2}{R}} \qquad \text{For } r < R$$

Potential Due to Group of Charges

The potential at a point due to "n" number of point charges is

$$V = \sum_{i=1}^{n} V_i = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_i}$$

where q_i is the ith point charge and r_i is the distance from the ith charge to the point of interest.

Example: Potential Due to Group of Charges

Example:

$$q_1 = +12\text{nC}, \quad q_3 = +31\text{nC}$$

$$q_2 = -24\text{nC}, \quad q_4 = +17\text{nC}$$

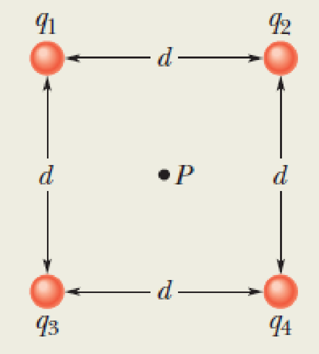


Figure 3: Figure adopted from HRW

Example: Potential Due to Group of Charges

What is electric potential at P if $d = 1.3m$?

$$x = \frac{\sqrt{2}}{2}d = \frac{d}{\sqrt{2}}$$

where x is the distance from each charge to the point P.

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{r} + \frac{q_2}{r} + \frac{q_3}{r} + \frac{q_4}{r}\right]$$

$$V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\sqrt{2}}{d}(36 \times 10^{-9})$$

$$\boxed{V = 350 volt}$$

Potential Energy of a System of Point Charges

For many point charges the energy will be the sum over all pairs of point charges. We know that for only two charges the PE is

$$U = \frac{kq_1 q_2}{r}$$

For many charges

$$U = \sum_{\text{all pairs}} \frac{kq_i q_j}{r_{ij}}$$

Example: Potential Energy of a System of Point Charges

$$q_1 = +q, \quad q_2 = -4q, \quad q_3 = 2q, \quad q = 150\text{nC}$$

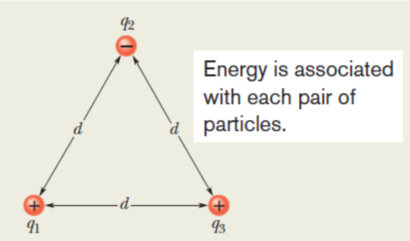


Figure 4: Figure adopted from HRW

Example: Electric potential energy of a system of point charges

$$U = \frac{kq_1q_2}{d} + \frac{kq_1q_3}{d} + \frac{kq_2q_3}{d}$$

$$U = \frac{k}{d}\left[q(-4q) + q(2q) + (-4q)(2q)\right]$$

$$U = \frac{k}{d}[-4q^2 + 2q^2 - 8q^2] = -\frac{k}{d}10q^2$$

$$U = -\frac{(9 \times 10^9)(10)(150 \times 10^{-9})^2}{(12 \times 10^{-2})^2}$$

$$U = \boxed{-1.7 \times 10^{-2}\ \text{J}} = -17\ \text{mJ}$$

Potential of Continuous Charge (limited charge distribution)

suppose we have a semi circle on which charge Q is distributed uniformly as shown in the figure below

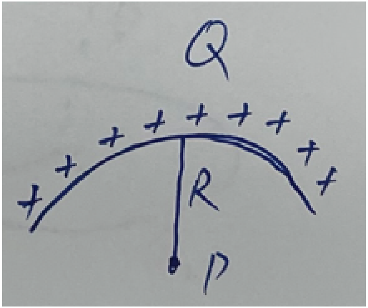


Figure 5: Figure adopted from HRW

Potential of Continuous Charge (limited charge distribution)

What is potential at P?

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{R}$$

$$V = \int dV = \frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{R}\int dq = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{R}$$

$$\boxed{V = \frac{kQ}{R}}$$

Potential of Continuous Charge (limited charge distribution)

Example In this example the arc is not a semi circle, instead it makes an angle 120^0 at the center. The Radius is $R = 3.71cm$ and the total charge on the arc is $Q = -25.6pC$. What will be electric potential at point P?

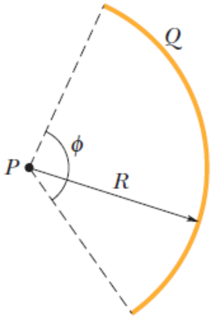


Figure 6: Figure adopted from HRW

Potential of Continuous Charge (limited charge distribution)

$$V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{R}$$

$$Q = -2.56 \times 10^{-11}\ \text{C}, \quad R = 0.0371\ \text{m}$$

$$V = \frac{(9 \times 10^9)(-2.56 \times 10^{-11})}{0.0371} = \boxed{-6.2\ \text{volts}}$$

The angle does not matter as long as the entire charge distribution is at the same distance from point P.

Potential of Continuous Charge (limited charge distribution)

Practice Problem

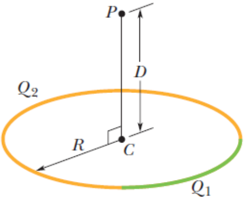


Figure 7: Figure adopted from HRW

What is V at point p & C in the figure above?

$$R = 8.2 \text{ cm}, \quad D = 6.71 \text{ cm}$$

$$Q_1 = 4.2 \text{ pC}, \quad Q_2 = -6Q_1$$