Chapter 25 - Capacitance

Capacitance

Capacitance: The ability to store charge.
When parallel plates are used to store charge (or energy), then we say the plates form a capacitor. The figure below shows a capacitor. When the switch S is closed, then charge start building on the plates.

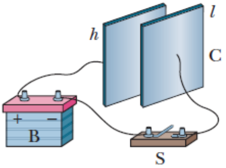

Figure 1: Figure adopted from HRW

Capacitance

The charge on the plates is directly proportional to the voltage of the battery.

$$Q \propto V$$

$$Q = CV$$

C proportionality constant is and known as capacitance of the capacitor. It depends on the geometry of the capacitor and is measured in Farads.

$$1\text{Farads} = \frac{1\text{Coulombs}}{1\text{Volt}}$$

A simple parallel plate capacitor has two parallel plates with area "A" and are separated by distance "d".

Charging a Capacitor

To charge a capacitor, we place it in an electric circuit (electric circuit is a path through which charge can flow), as shown in the figure below.

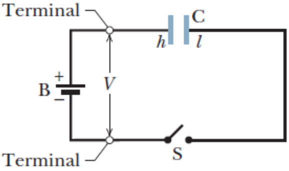

Figure 2: Figure adopted from HRW

The battery maintains potential difference V between its terminals.

Charging a Capacitor

When the circuit is complete, electrons are driven through the wires by an electric field that the battery sets up in the wires. The field drives electrons from capacitor plate h to the positive terminal of the battery, thus, plate h, losing electrons, becomes positively charged.
The field drives just as many electrons from the negative terminal of the battery to capacitor plate l, thus, plate l, gaining electrons, becomes negatively charged just as much as plate h, losing electrons becomes positively charged.
Thus the capacitor is then said to be fully charged, with a potential difference V between it's plates and charge Q on each plate.

$$Q = CV$$

Capacitance of a Parallel Plate Capacitor

$$V = \vec{E} \cdot \vec{d}$$

$$V = \left(\frac{\sigma}{\varepsilon_0}\right) d = \frac{Q}{A\varepsilon_0} d \Rightarrow V = \frac{Qd}{A\varepsilon_0} \Rightarrow Q = \frac{A\varepsilon_0}{d} V$$

$$\Rightarrow C = \frac{A\varepsilon_0}{d}$$

We can see the C depends on:

- Geometry of the plates
- Material between the plates (ε_0)

Capacitance of a Cylindrical Capacitor

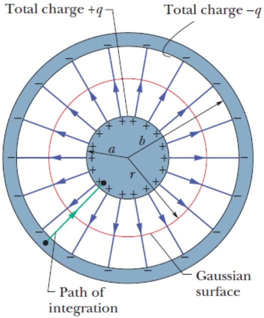

Figure 3: Figure adopted from HRW

$$E(2\pi rL) = \frac{q}{\varepsilon_0} \Rightarrow E = \frac{q}{2\pi\varepsilon_0 rL}$$

Capacitance of a Cylindrical Capacitor

$$V = -\int_b^a \vec{E} \cdot ds = -\frac{q}{2\pi\varepsilon_0 L}\int_b^a \frac{dr}{r} = \frac{q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right)$$

$$q = \frac{2\pi\varepsilon_0 L}{\ln(b/a)}V \Rightarrow C = \frac{2\pi\varepsilon_0 L}{\ln(b/a)}$$

Again, C depends only on geometrical factor of the capacitor.

A Spherical Capacitor

The above figure can serve as Spherical capacitor as well. So the above figure will now represent as two concentric spherical shell

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

$$V = -\int_b^a \vec{E} ds = -\frac{q}{4\pi\varepsilon_0}\int_b^a \frac{dr}{r^2} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{b-a}{ab}\right) \Rightarrow q = \frac{4\pi\varepsilon_0 ab}{b-a}V \Rightarrow \boxed{C = \frac{4\pi\varepsilon_0 ab}{b-a}}$$

Depends only on geometry factors.

An Isolated Sphere

We can find the capacitance of an isolated sphere by assuming the other shell is at infinity.

$$C = 4\pi\varepsilon_0 \frac{a}{1 - a/b} = 4\pi\varepsilon_0 R \Rightarrow \boxed{C = 4\pi\varepsilon_0 R}$$

The following figure shows some basic circuit elements.

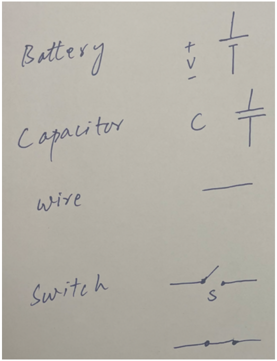

Figure 4: Figure adopted from HRW

Series and Parallel combination of Circuit elements

Two basic ways to connect circuit elements together:

- in series
- in parallel

Capacitors in Parallel

The following figure shows a parallel combination of three capacitors.

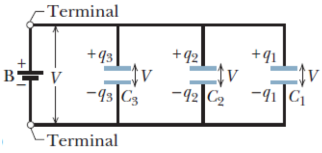

Figure 5: Figure adopted from HRW

We can see that they are all connected to the same terminal of the battery. That's why all the capacitors have the same voltage across them. The total charge will be:

$$Q = Q_1 + Q_2 + Q_3$$

Capacitors in Parallel

The charge stored across each of them will be

$$Q_1 = C_1 V, \quad Q_2 = C_2 V, \quad Q_3 = C_3 V$$

$$CV = C_1 V + C_2 V + C_3 V$$

$$C_{eq} = C_1 + C_2 + C_3$$

This means the above circuit can be replaced by the following simple circuit.

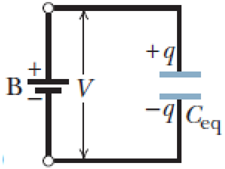

Figure 6: Figure adopted from HRW

Capacitors in Series

The following figure shows a series combination of capacitors. In series, there is a voltage drop across each capacitor so the voltage across each capacitor is not the same, but charge is the same on all the capacitors.

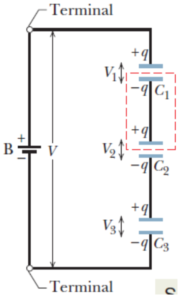

Figure 7: Figure adopted from HRW

Capacitors in Series

The battery voltage will be equal to the sum of voltage drops across each capacitor.

$$V = V_1 + V_2 + V_3$$

where

$$V_1 = \frac{Q}{C_1} \quad , V_2 = \frac{Q}{C_2} \quad , V_3 = \frac{Q}{C_3}$$

$$V = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3}$$

$$\frac{Q}{C} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3}$$

$$\boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}}$$

Capacitors in series

When there is a combination of capacitors in a circuit, we can replace that combination with an equivalent capacitor, that is, a single capacitor that has the same capacitance as the actual combination of capacitors.

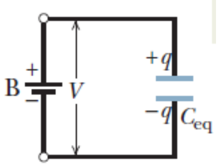

Figure 8: Figure adopted from HRW

Such replacement, we can simplify the circuit, affording easier solutions for unknown quantities of the circuit.

Example Problem

For the figure below: (a) Find C_{eq} ?
(b) If $V = 12.5V$ what is charge on C_1?

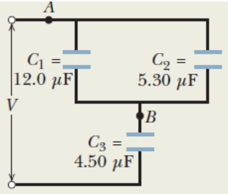

Figure 9: Figure adopted from HRW

$$C_1 = 12\,\mu F, \quad C_2 = 5.3\,\mu F, \quad C_3 = 4.5\,\mu F$$

Example Solution

(a) C1 and C2 are parallel because they are connected to the same terminals. (The total charge appears on C_3. The upper plate is charged $+Q$, and opposite C_3 plate and the top plates forward to the two plates toward C1 and C2. That is why C1 or C2 is in series with C3.)

$$C_{12} = C_1 + C_2 = 17.3\,\mu F$$

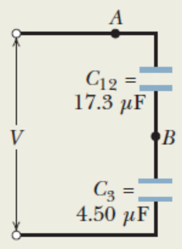

Figure 10: Figure adopted from HRW

Example Solution

$$\frac{1}{C_{eq}} = \frac{1}{C_{12}} + \frac{1}{C_3} = \frac{1}{17.3} + \frac{1}{4.5}$$

$$C_{eq} = \frac{(17.3)(4.5)}{4.5 + 17.3} = 3.57\,\mu F$$

(b) What is the charge on C_1?

$$Q = C_{eq} V = (3.57)(12.5) = 44.625\,\mu C$$

$$V_{12} = \frac{Q}{C_{12}} = \frac{44.625}{17.3} = 2.58\ V$$

This is the same potential across C_1 & C_2.

Example Solution

$$Q_1 = V_{12} C_1 = (2.58)(12 \times 10^{-6})$$

$$Q = 30.96 \mu C$$

Energy Stored in Capacitor

Charging a capacitor means moving charge from one plate of the capacitor to the other.
Let's suppose dq is some charge that is moved to the positive plate.
The work required to move that dq from the negative plate to the positive plate is:

$$dW = V\,dq, \quad V = \frac{q}{C}, \quad dW = \frac{q}{C}dq$$

Energy Stored in Capacitor

Total work required to charge the capacitor to charge Q:

$$W = \int dW = \int_0^Q \frac{q}{C} dq = \frac{1}{C} \int_0^Q q \, dq = \frac{1}{C} \cdot \frac{Q^2}{2} = \frac{Q^2}{2C}$$

$$W = \frac{Q^2}{2C}$$

The work done to charge the capacitor is stored as energy in the capacitor.
This work done is electrostatic potential energy between the field.

$$U = \frac{1}{2} \cdot \frac{Q^2}{C}$$

Energy Stored in Capacitor

Or

$$U = \frac{1}{2}CV^2$$

$$U = \frac{1}{2}(CV)^2 \cdot \frac{1}{C} = \frac{1}{2}CV^2$$

$$\boxed{U = \frac{1}{2}CV^2}$$

Energy Stored in a Parallel Plate Capacitor

$$U = \frac{1}{2} C V^2$$

$$C = \frac{A\varepsilon_0}{d}, \quad V = \vec{E} \cdot d$$

$$U = \frac{1}{2} \cdot \frac{A\varepsilon_0}{d} \cdot (\vec{E} \cdot d)^2 = \frac{1}{2} \varepsilon_0 E^2 (Ad)$$

Energy Stored in a Parallel Plate Capacitor

Since Ad: volume of the capacitor,

$$\frac{U}{Ad} = \frac{\text{Energy}}{\text{Volume}} = \frac{1}{2}\varepsilon_0 E^2$$

$$\text{Energy density} = u = \frac{U}{Ad} = \frac{1}{2}\varepsilon_0 E^2$$

$$\boxed{u = \frac{1}{2}\varepsilon_0 E^2}$$

Although we derive this for a parallel plate, it holds for any capacitor. This suggests that energy is stored in E-field.

=>E-field Contains Energy

Example: Isolated Conducting Sphere

The radius and charge on an isolated conducting sphere is given below.

$$R = 6.85\,\text{cm}, \quad q = 1.25\,\text{nC}$$

Find: (a) Energy Stored in the E-field ($U = ?$)

$$U = \frac{1}{2}CV^2 = \frac{1}{2} \cdot 4\pi\varepsilon_0 R \cdot \left(\frac{q}{4\pi\varepsilon_0 R}\right)^2$$

$$U = \frac{(1.25 \times 10^{-9})^2}{8 \cdot 3.14 \cdot 8.85 \times 10^{-12} \cdot 6.85 \times 10^{-2}}$$

$$\boxed{U = 103\,\text{nJ}}$$

Example: Isolated Conducting Sphere

(b) Energy density at the Surface of the Sphere

$$u = \frac{1}{2}\varepsilon_0 E^2$$

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{R^2} \quad \text{(at the surface)}$$

$$u = \frac{1}{2}\varepsilon_0 \cdot \frac{q^2}{16\pi^2\varepsilon_0^2 R^4} = \frac{q^2}{32\pi^2\varepsilon_0 R^4}$$

$$\boxed{u = 25.4\,\mu J/m^3}$$

Capacitor with a Dielectric

First time Michael Faraday (1837) checked the effect of introducing an insulating material inside a capacitor.
He found that the capacitance increases by a factor K, which is called the dielectric constant of the insulating material.

Air:	$K = 1.00054$
Paper:	$K = 3.5$
Water (20°C):	$K = 80.4$
Vacuum:	$K = 1$

Capacitor with a Dielectric

Since capacitance for any geometry can be written in the form:

$$C = \varepsilon_0 \mathscr{L}$$

Where $\mathscr{L}$ represents the geometry of the capacitor.
For parallel plate capacitor it is:

$$\mathscr{L} = \frac{A}{d}$$

So Faraday discovered that by completely filling the space with dielectric material, the capacitance becomes:

$$C = K\varepsilon_0 \mathscr{L} = KC_{air}$$

Where C_{air} is the value of capacitance with only air in between.

Effect of Dielectric

By comparing equations $C = \varepsilon_0 \mathscr{L}$ and $C = K\varepsilon_0 \mathscr{L}$, we see the effect of a dielectric can be summed up in more general terms:
"In a region completely filled by a dielectric material of dielectric constant K, all electrostatic equations containing the permittivity constant ε_0 are to be modified by replacing ε_0 with $K\varepsilon_0$."

Effect of Dielectric

Then:

$$E = \frac{1}{4\pi K\varepsilon_0} \cdot \frac{q}{r^2}$$

A point charge immersed in a dielectric — the electric field is reduced by a factor of K.

$$\text{Similarly,} \quad E = \frac{\sigma}{K\varepsilon_0} \quad \text{(field due to an infinite charged sheet)}$$