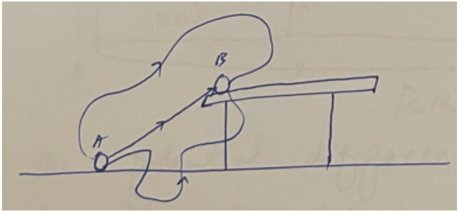
Chapter 27 - Circuits

Energy Conservation in Circuits

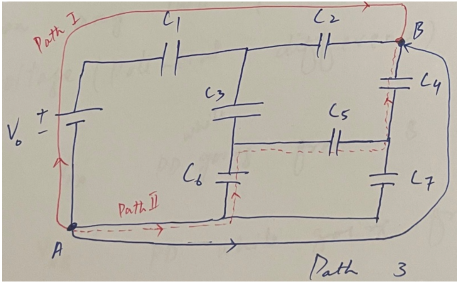
Energy is conserved, this means that the change in energy is path independent. If we start at any point A in a circuit, and finish at any other point B in the circuit, then the energy change from point 'A' to point B is independent of the path followed.

Mechanical Analogy:
As long as the ball starts at A and end at B, the same amount of work is done and the change in PE is the same.



Energy Conservation in Circuits

The same is true for electrical circuits. That is, the change in electric potential energy is the same going from point 'A' to 'B' while following different paths. From the figure we can see the the change in electric potential energy is the same while going from point A to B, through different paths. Since $V = \frac{U}{q}$, then the change in electric potential is also the same.



Energy Conservation in Circuits

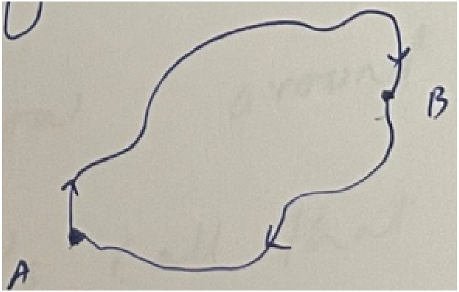
The potential difference between A & B V_{BA} for path I is

$$V_{BA} = V + V_1 + V_2$$
$$V_{BA} = V_6 + V_5 + V_4$$
$$V_{BA} = V_7 + V_4$$

According to conservation of energy, they should all be the same. If V_{BA} is path independent, then any two paths must have the same voltage (potential difference).

Energy Conservation in Circuits

Now, let's consider the following example.



Energy Conservation in Circuits

V_{BA} : PD while going from B to A

V_{AB} : PD while going from A to B

$V_{BA} = -V_{AB}$

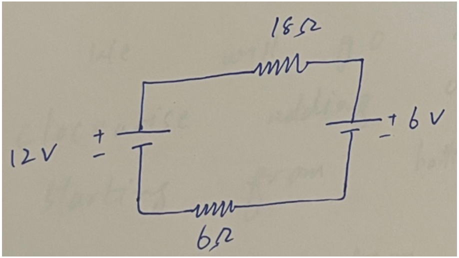
$$\boxed{V_{BA} + V_{AB} = 0}$$

Voltage around any closed path is zero.

Kirchhoff's Voltage Law (KVL)

Sum of voltages around any closed loop in any circuit is zero.

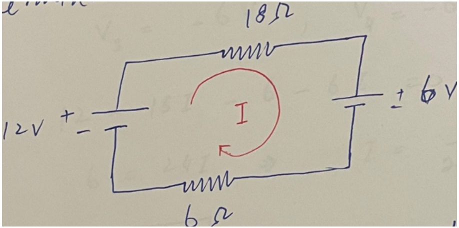
Example: Kirchhoff's Voltage Law (KVL)



What is I in the circuit (figure below)?

Example: Kirchhoff's Voltage Law (KVL)

Solution Only one current can flow around a single-loop circuit. Let's call that current I (figure below). Conventional current (positive charges) goes from higher potential to lower potential. Positive is higher potential than negative terminal.



Example: Kirchhoff's Voltage Law (KVL)

We will go around the circuit clockwise adding up voltages starting from bottom left-hand corner.
If we go from $-$ potential to $+$ potential then there is a gain in potential and we take it as positive.

$$V_1 = 12\,\mathrm{V}$$

Then voltage drop across $18\,\Omega$

$$V_2 = -18I, \quad V_3 = -6V, \quad V_4 = -6I$$

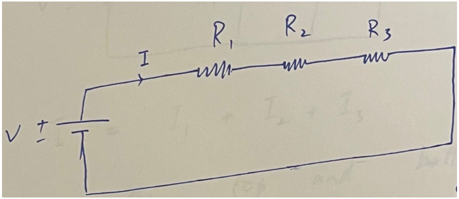
$$12 - 18I - 6 - 6I = 0 \Rightarrow 6 = 24I \Rightarrow I = \frac{6}{24} = \frac{1}{4}\,\mathrm{A}$$

$$\boxed{I = \frac{1}{4}\,\mathrm{A}} \quad \text{(clockwise)}$$

Combining Resistors

Just like we combined capacitors from series and parallel combination to get an equivalent capacitor, we can get an equivalent resistor from a the series and parallel combination.

Resistors in Series



For components in series, same current must flow through all of them (since there is only one path of flow). There is a voltage drop across each resistor and the total voltage is equal to the sum of the voltages across each resistor.

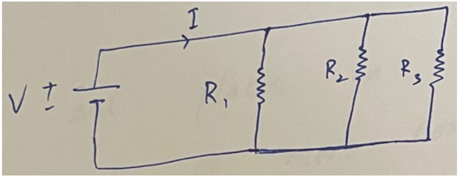
$$V = V_1 + V_2 + V_3$$

$$IR = IR_1 + IR_2 + IR_3$$

$$\boxed{R = R_1 + R_2 + R_3}$$

Resistors in Parallel

If the resistors are in parallel (as shown below), then there are multiple paths to the flow of current and different current flow through each branch.



The total current from the battery will be equal to the sum of the current in each branch.

$$I = I_1 + I_2 + I_3$$

Resistors in Parallel

Since the top and bottom of the resistors are connected with the same battery, they all have the same potential.

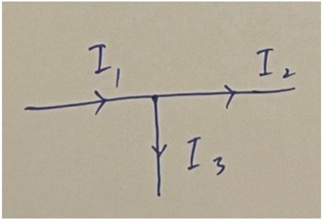
$$I_1 = \frac{V}{R_1}, \quad I_2 = \frac{V}{R_2}, \quad I_3 = \frac{V}{R_3}$$

$$\frac{V}{R} = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3}$$

$$\boxed{\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}$$

Kirchhoff's current law (KCL)

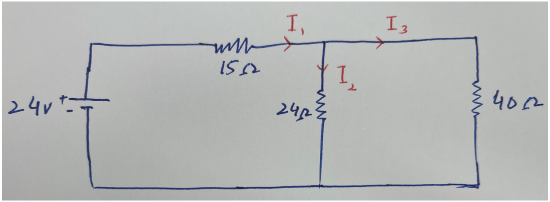
Nodes are places in the circuit where 3 or more components meet.
And since charge is conserved, then any current that flows into a node, that same current must flow out of the node.
KCL: Sum of currents into a node must equal sum of currents flowing out of that node.



$$I_1 = I_2 + I_3$$

Example: Applying Kirchhoff's Laws

Find I_1, I_2 and I_3 in the following circuit.



Example: Applying Kirchhoff's Laws

Solution:

KVL for loop 1:

$$24 - 15I_1 - 24I_2 = 0$$

$$\boxed{8 - 5I_1 - 8I_2 = 0} \quad \text{Equation 1}$$

KVL for loop 2:

$$24I_2 - 40I_3 = 0$$

$$I_3 = \frac{24}{40} I_2$$

$$\boxed{I_3 = \frac{3}{5} I_2} \quad \text{Equation 2}$$

KCL:

$$\boxed{I_1 = I_2 + I_3} \quad \text{Equation 3}$$

Example: Applying Kirchhoff's Laws

From Equation 2 and 3:

$$\boxed{I_1 = I_2 + \frac{3}{5} I_2 = \frac{8}{5} I_2} \quad \text{Equation 4}$$

Putting this in Equation 1:

$$8 - 5\left(\frac{8}{5} I_2\right) - 8I_2 = 0 \Rightarrow 8 - 8I_2 - 8I_2 = 0 \Rightarrow 8 - 16I_2 = 0$$

$$\boxed{I_2 = \frac{1}{2} A}$$

Example:

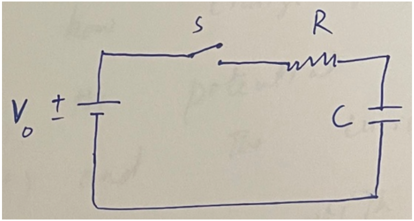
Putting I_2 in Equation 4 and Equation 2 to find I_1 and I_3 respectively.

$$\boxed{I_1 = \frac{4}{5}\text{A}}, \quad \boxed{I_3 = \frac{3}{10}\text{A}}$$

RC Circuits

Now we deal with the circuits in which both R & C are present. In this case the current is not constant in the circuit but vary with time.

Charging a Capacitor



As we complete the circuit (by closing the switch in the circuit below) the current starts to flow and it begin charging the capacitor. This current increases the charge q on the plates and also the potential difference across the capacitor

$$V_c = \frac{q}{C}$$

Where V_c is voltage across the capacitor. When the potential difference across the capacitor (V_c) becomes equal to the voltage of the battery (V_0), the current is zero. and the final equilibrium charge on the plates is

$$q_0 = CV_0$$

Charging a Capacitor

Here we want to examine the charging process. In particular we want to know how the potential difference $V(t)$ and the current $I(t)$ in the circuit vary with time during the charging process.
KVL

$$V_0 - IR - \frac{q}{C} = 0$$

$$V_0 - IR = \frac{q}{C}$$

$$V_0 = IR + \frac{q}{C}$$

To solve this for $I(t)$, let's take derivative of this w.r.t time

$$\frac{dV_0}{dt} = R\frac{dI}{dt} + \frac{1}{C}\frac{dq}{dt}$$

Charging a Capacitor

$$0 = R\frac{dI}{dt} + \frac{1}{C}I$$

$$\frac{dI}{dt} = -\frac{1}{RC}I$$

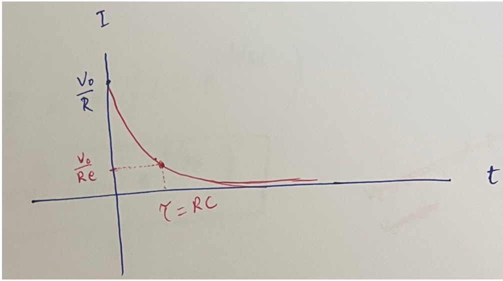
$$\int_{I_0}^{I} \frac{dI}{I} = -\frac{1}{RC}\int_0^t dt$$

$$\ln\left(\frac{I}{I_0}\right) = -\frac{1}{RC}t$$

$$I = I_0 e^{-t/RC}$$

$$\boxed{I(t) = \frac{V_0}{R} e^{-t/RC}}$$

This is the current in the circuit as a function of time, where $I_0 = \frac{V_0}{R}$.



Charging a Capacitor

$$V_R = IR = \frac{V_0}{R} e^{-t/RC} R$$

$$\boxed{V_R(t) = V_0 e^{-t/RC}}$$

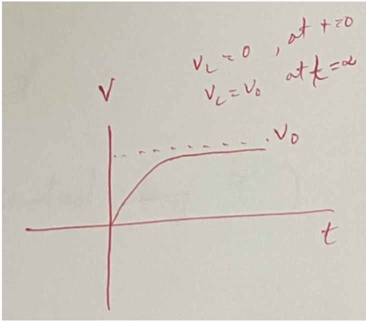
V_R is voltage across the resistor as a function of time.

Charging a Capacitor

$$V_c = V_0 - IR = V_0 - V_0 e^{-t/RC}$$

$$\boxed{V_c(t) = V_0(1 - e^{-t/RC})}$$

V_c is voltage across capacitor as a function of time. The following figure depict the behavior of V_c.

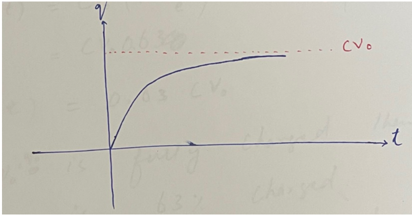


Charging a Capacitor

$$\frac{q(t)}{C} = V_0(1 - e^{-t/RC})$$

$$q(t) = CV_0(1 - e^{-t/RC})$$

$$\boxed{q(t) = q_0(1 - e^{-t/RC})}$$



Charging a Capacitor

The product RC has the dimensions of time and RC is called time constant (represented by τ)

$$\tau = RC$$

If $t = \tau$, this mean that in this duration of time, the capacitor has been charged up to 63%.

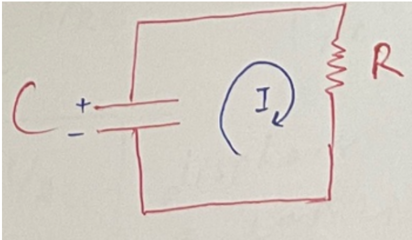
$$q(t) = q_0(1 - e^{-1}) = q_0(0.63) = 0.63q_0$$

$$\boxed{q(t) = 0.63q_0}$$

This means that after 1 time constant (RC) the capacitor is 63% charged. This means that the greater RC the greater will be the charging time.

Discharging a Capacitor

In the following figure an already charged capacitor is connected with a resistor. and once the switch is closed, a current I will start flowing in the circuit.



Discharging a Capacitor

$$\frac{q}{C} + IR = 0$$

$$q = -CR\frac{dq}{dt} \Rightarrow \frac{dq}{q} = -\frac{1}{RC}dt \Rightarrow \int_{q_0}^{q} \frac{dq}{q} = -\frac{1}{RC}\int_0^t dt$$

$$\ln\left(\frac{q}{q_0}\right) = -\frac{t}{RC}$$

$$\boxed{q(t) = q_0 e^{-t/RC}}$$

This is the charge on the capacitor as a function of time.

Discharging a Capacitor

$$\frac{dq}{dt} = I(t) = q_0 \left(-\frac{1}{RC}\right) e^{-t/RC}$$

$$I(t) = -\left(\frac{q_0}{RC}\right) e^{-t/RC}$$

$$\boxed{I(t) = -I_0 e^{-t/RC}}$$

where $I_0 = \left(\frac{q_0}{RC}\right)$ is the initial current in the circuit. The minus sign in the equation means that the charge on the capacitor is decreasing.

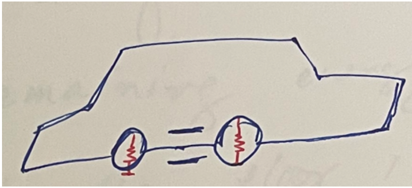
Example

A car rolls along a pavement, electron moves from the pavement and then onto the car body. The car store this excess charge and the potential energy in such a way that as if the car is one plate of a capacitor and the pavement another.
When the car stops it can discharge through the tires which behave like resistors. If a conducting object comes within a few centimeter before the car is discharged, the remaining energy can be suddenly transferred to a spark between the car and the object.
Suppose that object is a fuel dispenser then the spark can start a fire.
Suppose if the remaining energy between the car and the floor is less than

$$U = 50\, mJ$$

then there will be no spark and there will be no fire.

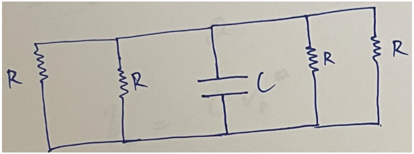
Example



How much time does the car take to discharge through the tires to drop the energy below 50 mJ?

The car four tires can be represented as four resistors connected in parallel.

Example



Given data:

$$V_0 = 30\,kV, \quad C = 500\,pF, \quad R_{\text{tire}} = 100\,G\Omega$$

Since the resistors are in parallel:

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R} + \frac{1}{R} + \frac{1}{R} + \frac{1}{R} = \frac{4}{R}$$

$$R_{\text{eq}} = \frac{R}{4}$$

$$U = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2C}(q_0 e^{-t/RC})^2$$

Example

$$U = \frac{1}{2C} q_0^2 e^{-2t/RC}$$

$$2CU = q_0^2 e^{-2t/RC}$$

$$e^{-2t/RC} = \frac{2CU}{q_0^2}$$

$$-\frac{2t}{RC} = \ln\left(\frac{2CU}{q_0^2}\right)$$

$$t = -\frac{RC}{2} \ln\left(\frac{2CU}{q_0^2}\right)$$

$$q_0 = CV_0$$

Example

$$t = -\frac{RC}{2} \ln\left(\frac{2U}{CV_0^2}\right)$$

$$t = -\frac{(25 \times 10^9)(500 \times 10^{-12})}{2} \ln\left(\frac{2(50 \times 10^{-3})}{(500 \times 10^{-12})(30 \times 10^3)^2}\right)$$

$$\boxed{t = 9.4\,\text{sec}}$$