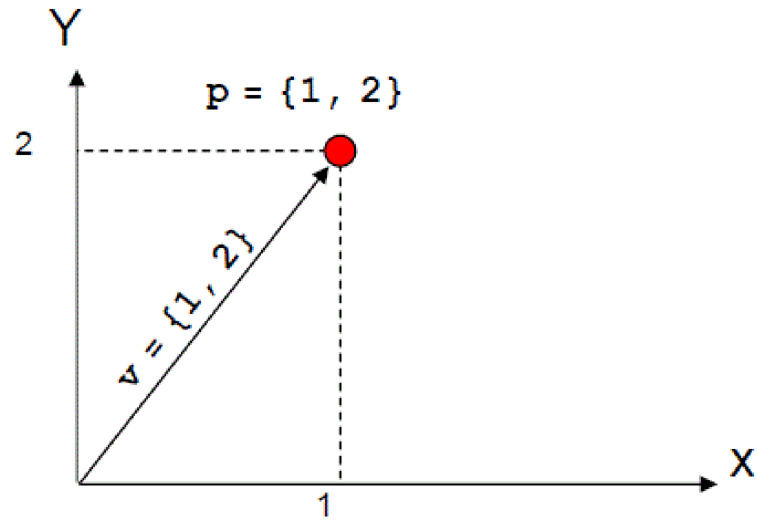
CSC 321 Computer Graphics

Points, Vectors, and Shapes

Points and Vectors

- Same representation

  `{x, y} (or {x, y, z})`



- Different meaning:
  - Point: a fixed **location** (relative to {0,0} or {0,0,0})
    - Coordinates change as location changes
  - Vector: a **direction** and **length**
    - Coordinates do not change as location changes

Points and Vectors

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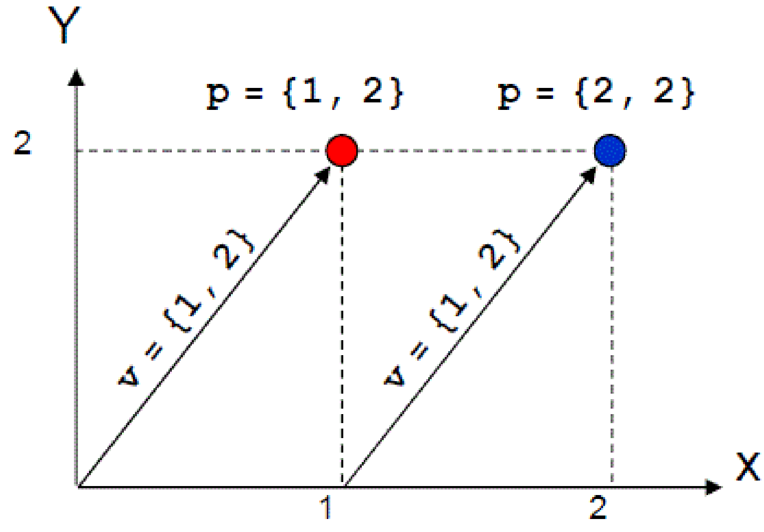
  `{x, y} (or {x, y, z})`



- Different meaning:
  - Point: a fixed **location** (relative to {0,0} or {0,0,0})
    - Coordinates change as location changes
  - Vector: a **direction** and **length**
    - Coordinates do not change as location changes

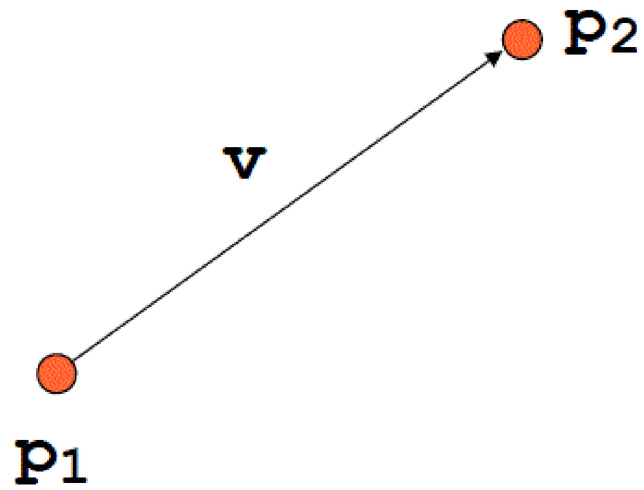
Point Operations

- Subtraction
  - Result is a vector

$$p_2 - p_1 = v = \{p_{2_x} - p_{1_x}, p_{2_y} - p_{1_y}\}$$

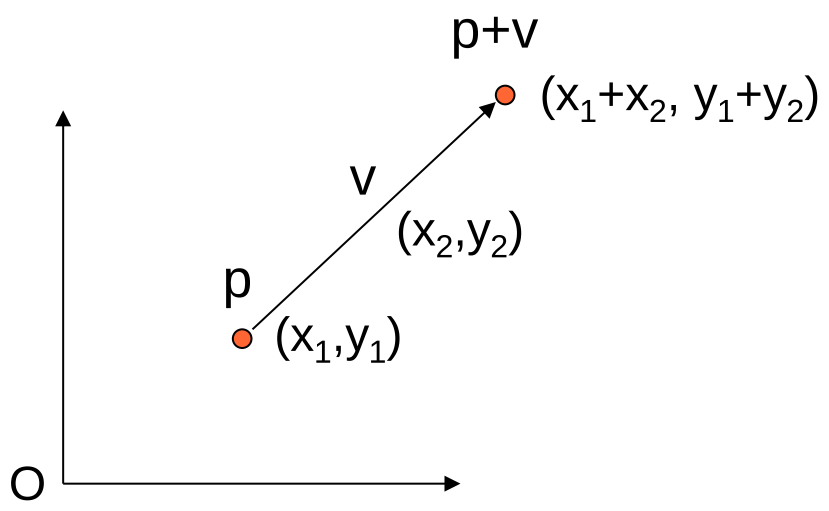
- Addition with a vector
  - Result is a point

$$p_1 + v = p_2 = \{p_{1_x} + v_x, p_{1_y} + v_y\}$$



Point Operations

- Addition with a vector
  - Resulting location does not change with the origin



Point Operations

- Addition with a vector
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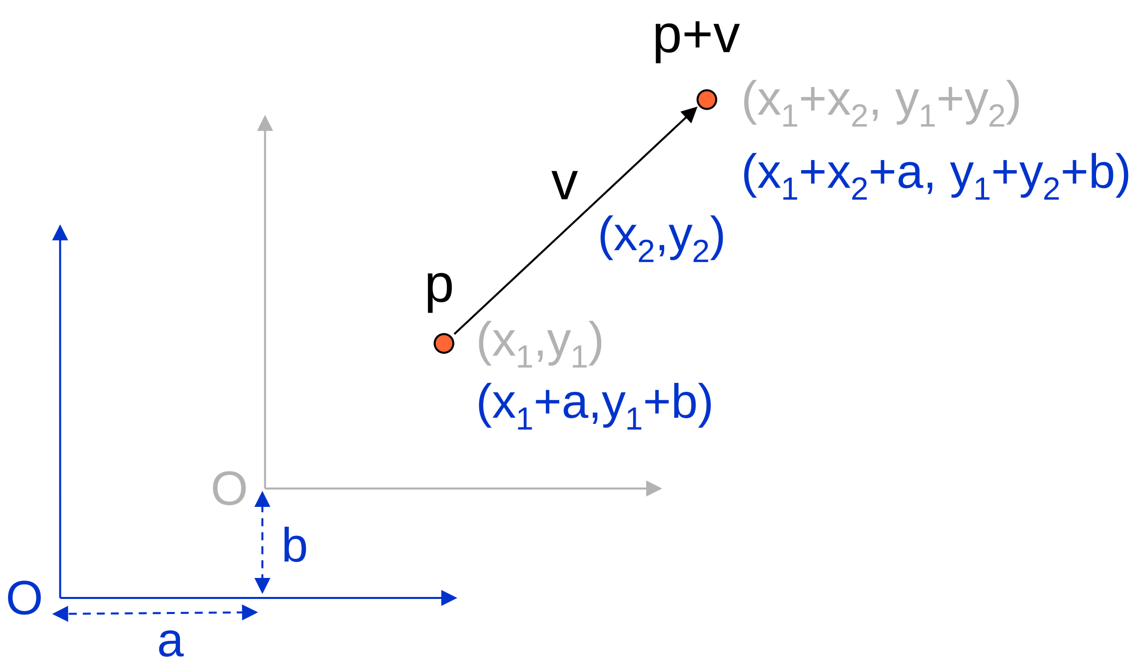
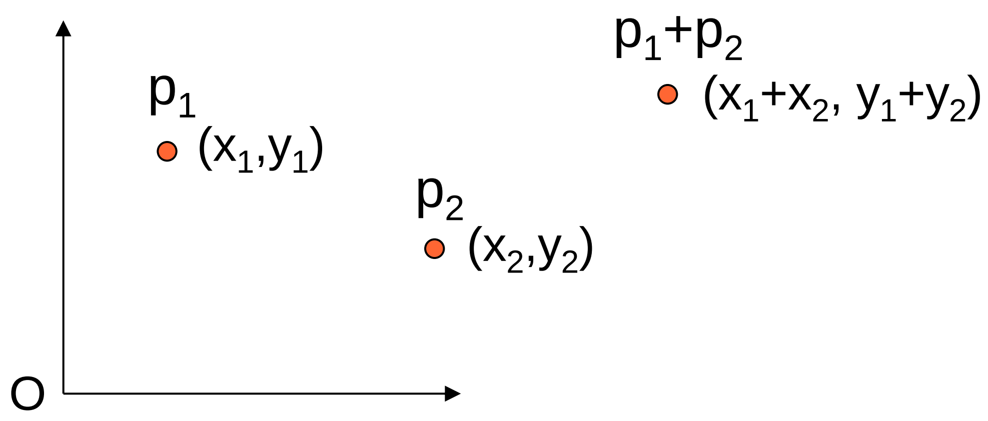


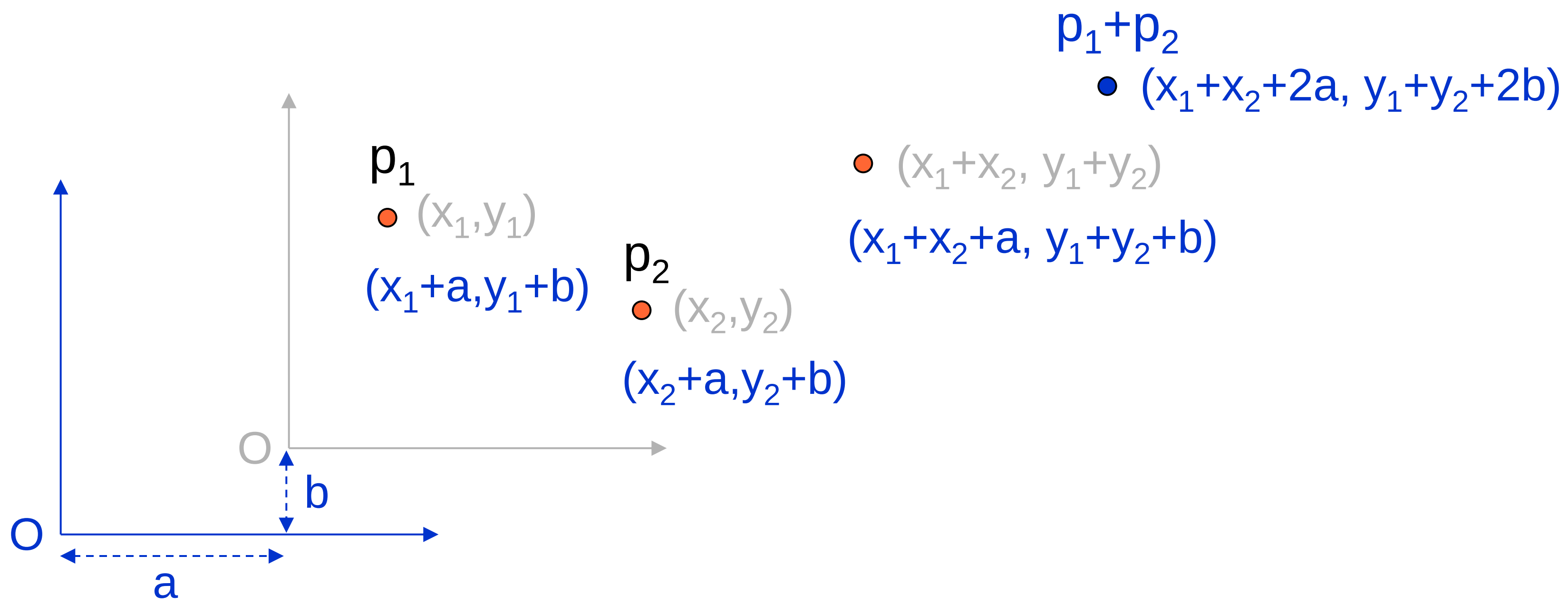
Point Operations

- Can two points add?



Point Operations

- Can two points add?
  - In general, **no**: result is dependent on where the origin is
    - But there are exceptions; will discuss later



Vector Operations

- Addition/Subtraction
  - Result is a vector

    $$v_1 \pm v_2 = \left\{ v_{1_x} \pm v_{2_x},\ v_{1_y} \pm v_{2_y} \right\}$$

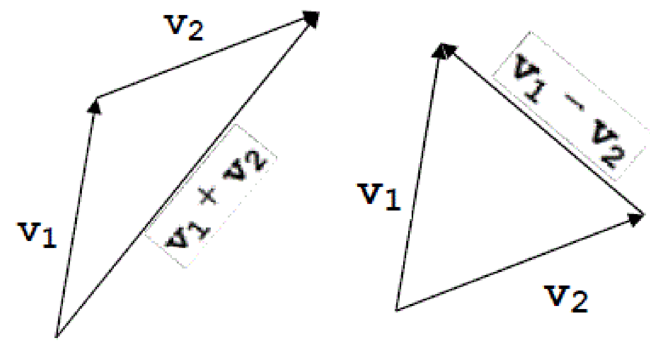
- Scaling by a scalar
  - Result is a vector

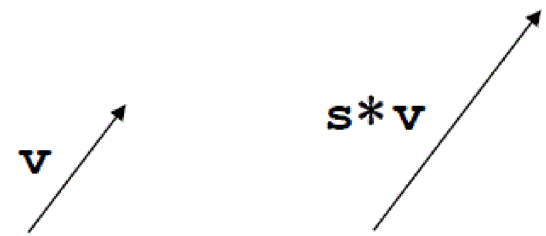
    $$s * v = \{ s * v_x,\ s * v_y \}$$

- Magnitude
  - Result is a scalar

    $$|\ v\ | = \sqrt{v_x^2 + v_y^2}$$

    - A unit vector: $|\ v\ | = 1$
    - To make a unit vector (normalization): $\dfrac{v}{|\ v\ |}$



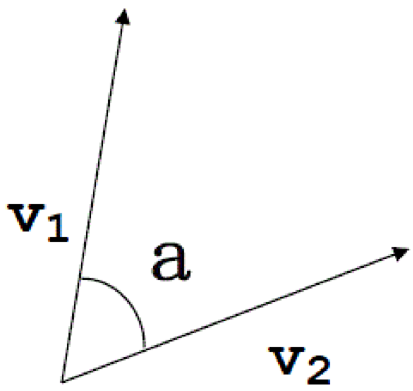


Vector Operations

- Dot product
  - Result is a scalar

    $\mathbf{v}_1 \cdot \mathbf{v}_2 = |\mathbf{v}_1||\mathbf{v}_2| \cos[\alpha]$

  - In coordinates (simple!)
    - 2D: $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_{1_x} * \mathbf{v}_{2_x} + \mathbf{v}_{1_y} * \mathbf{v}_{2_y}$
    - 3D: $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_{1_x} * \mathbf{v}_{2_x} + \mathbf{v}_{1_y} * \mathbf{v}_{2_y} + \mathbf{v}_{1_z} * \mathbf{v}_{2_z}$
    - Matrix product between a row and a column vector

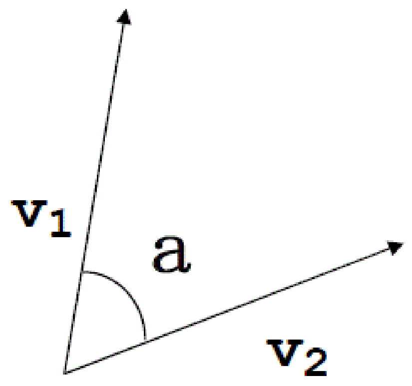


Vector Operations

- Uses of dot products
  - Angle between vectors:

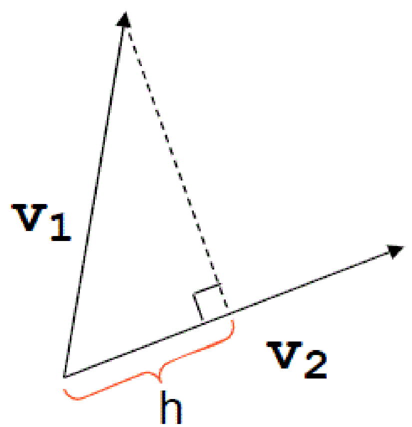
$$\alpha = \mathrm{ArcCos}\left[\frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|\mathbf{v}_1| * |\mathbf{v}_2|}\right]$$

    - Orthogonal: $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$



  - Projected length of $\mathbf{v}_1$ onto $\mathbf{v}_2$:

$$h = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|\mathbf{v}_2|}$$

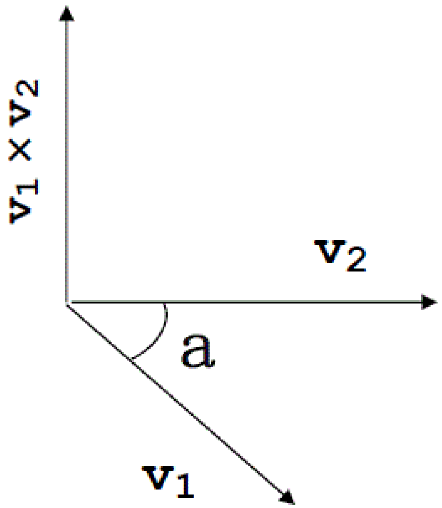


Vector Operations

- Cross product (in 3D)
  - Result is another 3D vector
    - Direction: Normal to the plane where both vectors lie (right-hand rule)
    - Magnitude: $|\, v_1 \times v_2 \,| = |\, v_1 \,|\,|\, v_2 \,| \, \mathrm{Sin}[\alpha]$
  - In coordinates:
    - Determinant of a matrix:

$$\begin{pmatrix} i & j & k \\ v1_x & v1_y & v1_z \\ v2_x & v2_y & v2_z \end{pmatrix}$$

$$v_1 \times v_2 = \{ v1_y\, v2_z - v1_z\, v2_y,\ v1_z\, v2_x - v1_x\, v2_z,\ v1_x\, v2_y - v1_y\, v2_x \}$$

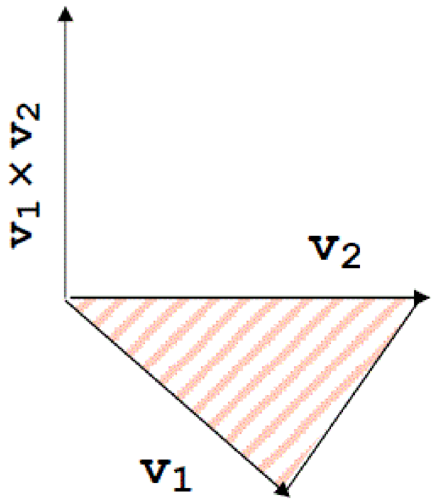


Vector Operations

- Uses of cross products
  - Getting the normal vector of the plane
    - E.g., the normal of a triangle formed by $\mathbf{v}_1\,\mathbf{v}_2$
  - Computing area of the triangle formed by $\mathbf{v}_1\,\mathbf{v}_2$

$$\text{Area} = \frac{|\,\mathbf{v}_1 \times \mathbf{v}_2\,|}{2}$$

- Testing if vectors are parallel: $|\,\mathbf{v}_1 \times \mathbf{v}_2\,| = 0$

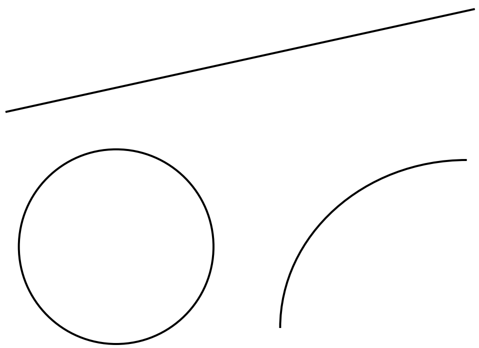


Vector Operations

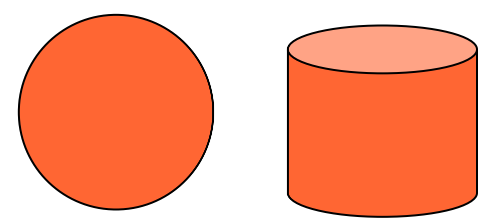
	Dot Product	Cross Product
Distributive?	$\mathbf{v} \cdot (\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{v} \cdot \mathbf{v}_1 + \mathbf{v} \cdot \mathbf{v}_2$	$\mathbf{v} \times (\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{v} \times \mathbf{v}_1 + \mathbf{v} \times \mathbf{v}_2$
Commutative?	$\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_2 \cdot \mathbf{v}_1$	$\mathbf{v}_1 \times \mathbf{v}_2 = -\mathbf{v}_2 \times \mathbf{v}_1$ (Sign change!)
Associative?	~~$\mathbf{v}_1 \cdot (\mathbf{v}_2 \cdot \mathbf{v}_3)$~~	$\mathbf{v}_1 \times (\mathbf{v}_2 \times \mathbf{v}_3) \neq (\mathbf{v}_1 \times \mathbf{v}_2) \times \mathbf{v}_3$

Shapes and Dimensions

- **0**-**d**imensional shape: point
  - No length or area
- **1**-**d**imensional shape: curve
  - Has non-zero “length”
  - Examples: line (segment), circle (arc), parabola, etc.



- **2**-**d**imensional shape: surface
  - Has non-zero “area”
  - Examples: filled triangle or quad, filled circle, surface of a cylinder, surface of a sphere, etc.

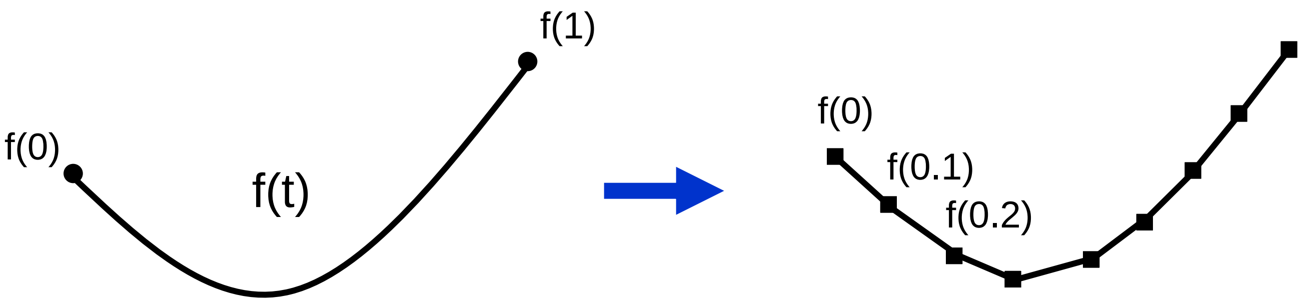


Tessellation

- Graphics cards are good at drawing tessellated elements
  - E.g., line segments, triangles, etc.

1D Tessellation

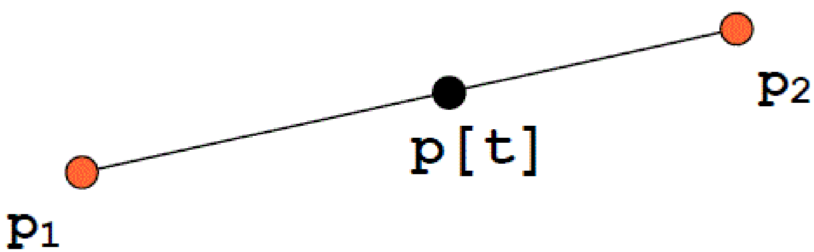
- Approximate a 1D curve shape by line segments
  - Define the curve as a function of one parameter
  - Generate samples on the curve at fixed intervals of the parameter
  - Connect successive samples by line segments



Parameterizing 1D Shapes

- A line segment:

$$p[t] = (1 - t)\, p_1 + t\, p_2 \qquad 0 \leq t \leq 1$$



Can Points Add? Sometimes.

- Linear interpolation (for two points)

  $$p = (1 - t)\, p_1 + t\, p_2$$

  - For any t, location of p is invariant to origin change
    - It is basically a point-and-vector addition:

      $$p = p_1 + t\, (p_2 - p_1)$$

Can Points Add? Sometimes.

- Affine combinations (for multiple points)

$$p = \sum_{i=1}^{n} t_i \, p_i , \quad \text{where} \sum_{i=1}^{n} t_i = 1$$

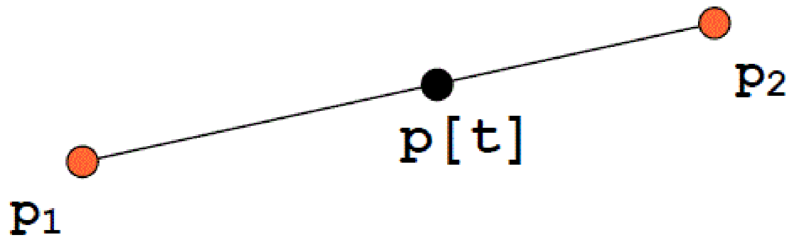
  - For any t_i, location of p is invariant to origin change
    - Again, a point-and-vector addition:

$$p = p_1 + \sum_{i=1}^{n} t_i \, (p_i - p_1)$$

Parameterizing 1D Shapes

- A line segment:

$$p[t] = (1 - t)\,p_1 + t\,p_2 \qquad 0 \le t \le 1$$

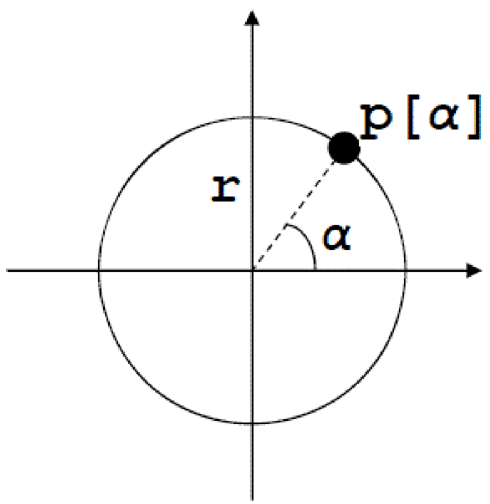


Parameterizing 1D Shapes

- Circle
  - Centered at origin with radius r

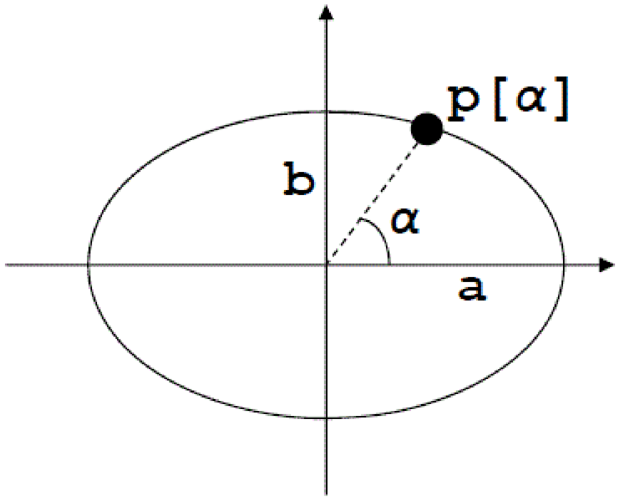
$$p[\alpha] = \{r\,\mathrm{Cos}[\alpha],\ r\,\mathrm{Sin}[\alpha]\}$$

$$0 \le \alpha < 2\pi$$



Parameterizing 1D Shapes

- Ellipse
  - Centered at origin with axes a, b

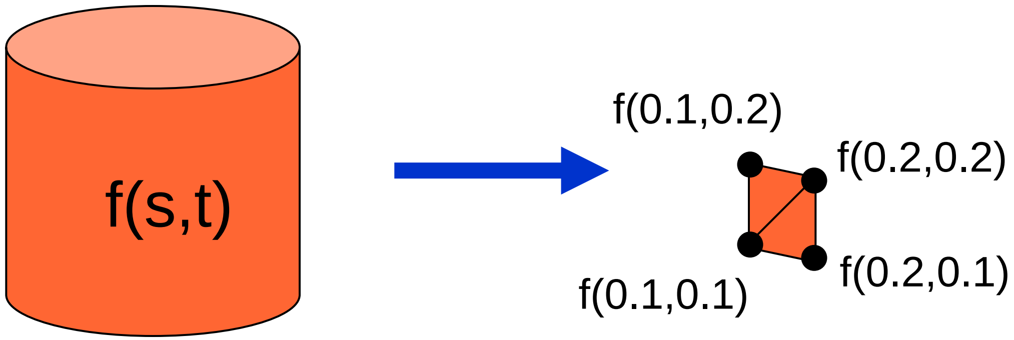


$$p[\alpha] = \{a\,\mathrm{Cos}[\alpha],\ b\,\mathrm{Sin}[\alpha]\}$$

$$0 \leq \alpha < 2\pi$$

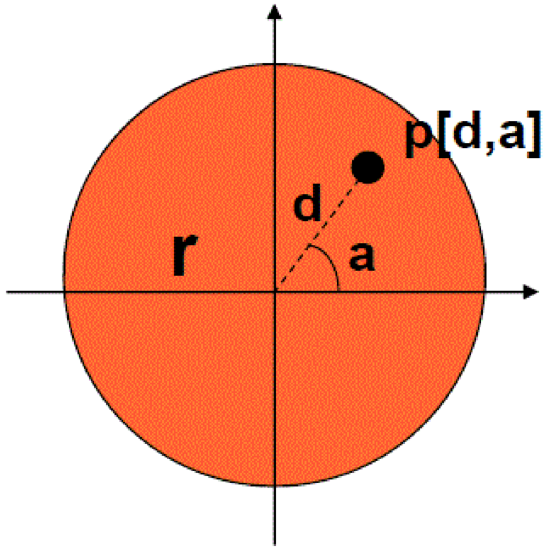
2D Tessellation

- Approximate a 2D surface shape by triangles
  - Define the surface as a function of two parameters
  - Generate samples at fixed intervals of both parameters
  - Connect samples by triangles



Parameterizing 2D Shapes

- Filled disk
  - Centered at origin with radius r

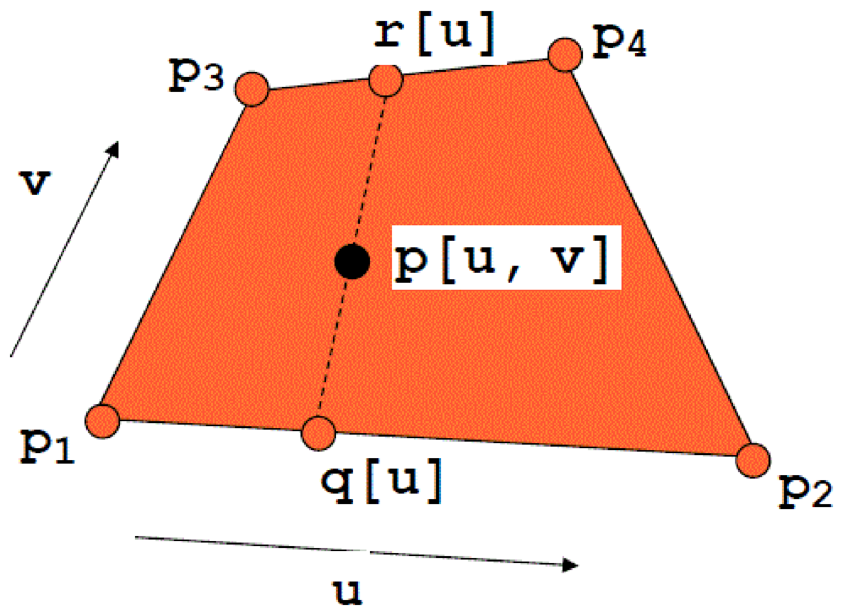


$$p[d, \alpha] = \{d\,\mathrm{Cos}[\alpha], d\,\mathrm{Sin}[\alpha]\}$$

$$0 \leq d \leq r,\ 0 \leq \alpha < 2\pi$$

Parameterizing 2D Shapes

- Filled quad



$$q[u] = (1 - u)\, p_1 + u\, p_2$$

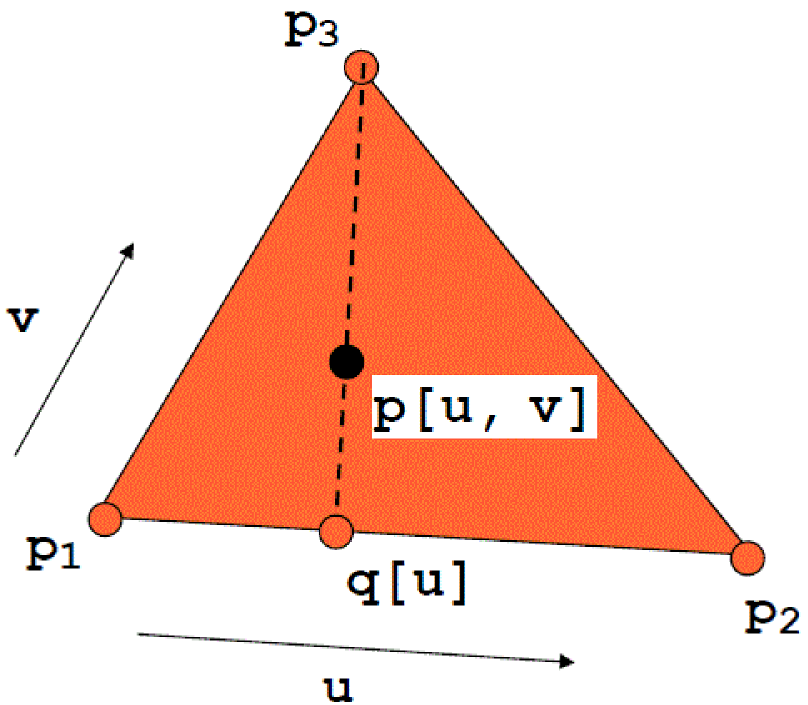
$$r[u] = (1 - u)\, p_3 + u\, p_4$$

$$p[u, v] = (1 - v)\, q[u] + v\, r[u]$$

$$0 \le u \le 1, \quad 0 \le v \le 1$$

Parameterizing 2D Shapes

- Filled triangle



$$q[u] = (1 - u)\, p_1 + u\, p_2$$

$$p[u, v] = (1 - v)\, q[u] + v\, p_3$$

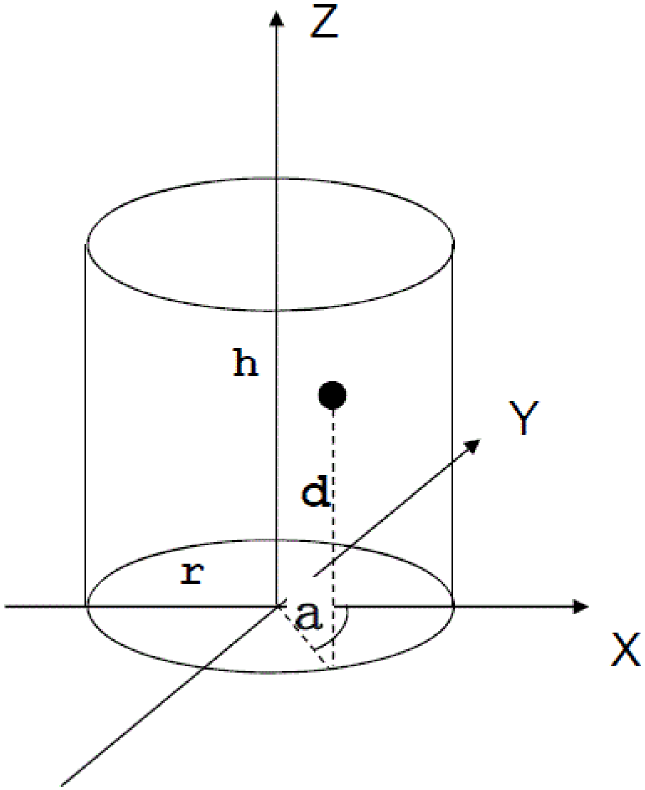
$$0 \le u \le 1, \quad 0 \le v \le 1$$

Parameterizing 2D Shapes

- Outer surface of a cylinder
  - Base centered at origin
  - Radius r, height h

$$p[d, \alpha] = \{r\,\mathrm{Cos}[\alpha], r\,\mathrm{Sin}[\alpha], d\}$$

$$0 \le d \le h,\ 0 \le \alpha < 2\pi$$



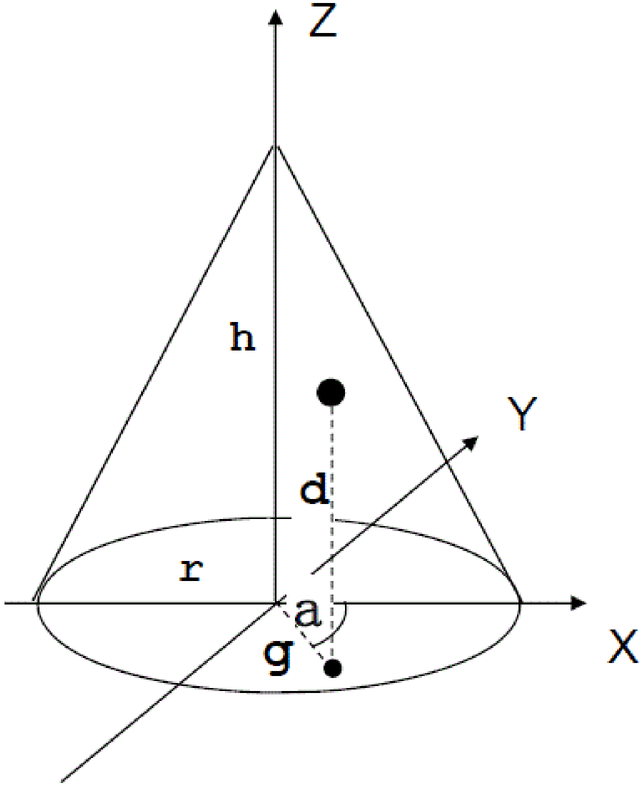
Parameterizing 2D Shapes

- Cone surface
  - Base centered at origin
  - Radius r, height h

$$p[d, \alpha] = \{g \operatorname{Cos}[\alpha], g \operatorname{Sin}[\alpha], d\}$$

$$g = \frac{r\,(h - d)}{h}$$

$$0 \leq d \leq h,\ 0 \leq \alpha < 2\pi$$

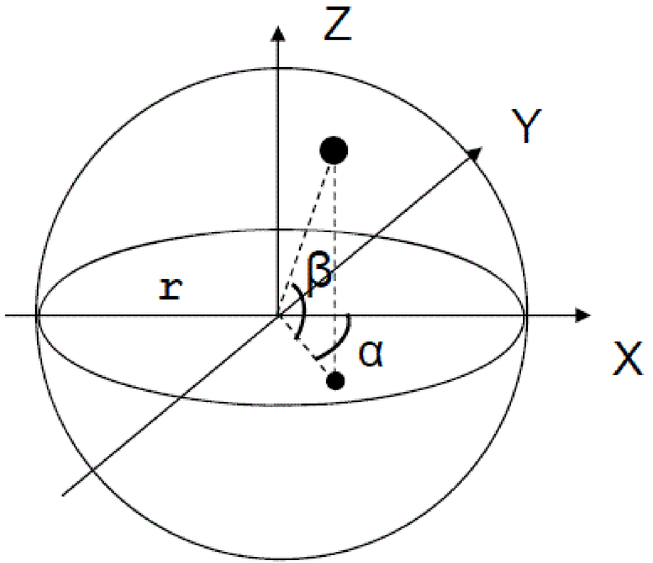


Parameterizing 2D Shapes

- Sphere surface
  - Centered at origin with radius r

$$p[\alpha, \beta] = \{r\,\mathrm{Cos}[\beta]\,\mathrm{Cos}[\alpha],\ r\,\mathrm{Cos}[\beta]\,\mathrm{Sin}[\alpha],\ r\,\mathrm{Sin}[\beta]\}$$

$$0 \le \alpha < 2\pi,\ \frac{-\pi}{2} \le \beta \le \frac{\pi}{2}$$



  - Not the best parameterization…