Physics - 2

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Simple Harmonic Motion

Any motion that repeat itself at regular intervals is called **periodic motion** or **harmonic motion**.

However, **Simple Harmonic Motion** is a specific type of periodic motion where the restoring force is directly proportional to the displacement from the equilibrium position and acts in opposite direction.

So, all Simple Harmonic Motion (SHM) are periodic, but not all periodic are SHM.

The figure below show a SHM

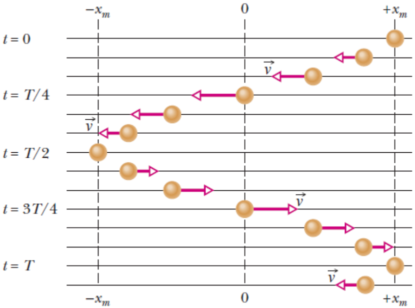


Figure 1: Figure Adopted from HRW

Simple Harmonic Motion

An important property of SHM is **Frequency**.
Frequency is the number of oscillations per second. It is represented by f, and it's unit is hertz (Hz).

$$1\ hertz = 1\ Hz = 1\ cycle/second = 1\ s^{-1} \tag{1}$$

Looking at the unit, we can define a new quantity of the SHM called **Time Period**.
Time Period is the time required to complete one Oscillation.

$$T = \frac{1}{f}$$

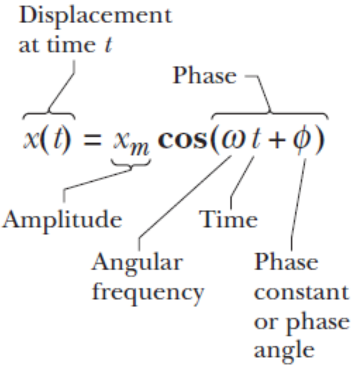
Simple Harmonic Motion

For a simple harmonic motion the displacement (distance from the equilibrium position) at any time t is given by

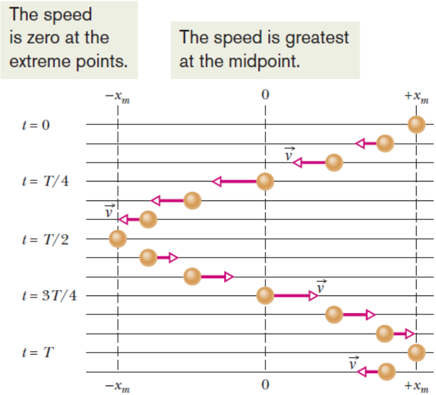
$$\boxed{x(t) = x_m cos(\omega t + \phi)}$$

$x(t)$ = displacement at any given time
x_m = Amplitude = Maximum displacement from the equilibrium (mean) position
ϕ = Phase constant = Information about the starting point of the SHM
ω = Angular frequency = How fast or slow the SHM is

Simple Harmonic Motion



Simple Harmonic Motion



Relating ω, f and T :

For one complete oscillation

$$x_m cost\ \omega t = x_m cos\ \omega(t + T)$$

Also the cosine function repeat itself after every 2π *rad*

$$x_m cos\ \omega t = x_m cos(\omega t + 2\pi)$$
$$\omega(t + T) = \omega t + 2\pi$$
$$\omega T = 2\pi$$

$$\boxed{\omega = \frac{2\pi}{T} = 2\pi f}$$

unit of ω is rad/s

Simple Harmonic Motion

Let's plot

$$x(t) = x_m cos(\omega t + \phi)$$

Consider $\phi = 0$

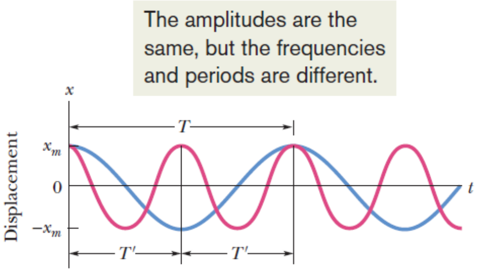


Figure 2: Figure Adopted from HRW

The Velocity of SHM:

$$v(t) = \frac{dx(t)}{dt} = \frac{d}{dt}[x_m \; cos \; (\omega t + \phi)]$$

$$\boxed{v(t) = -\omega x_m sin \; (\omega t + \phi)}$$

velocity amplitude $= v_m = \omega x_m$

The Acceleration of SHM:

$$a(t) = \frac{dv(t)}{dt} = \frac{d}{dt}[-\omega x_m sin(\omega t + \phi)]$$

$$\boxed{a(t) = -\omega^2 x_m cos \; (\omega t + \phi)}$$

acceleration amplitude $= a_m = \omega^2 x_m$

From the above equation

$$\boxed{a(t) = -\omega^2 x(t)}$$

This is the definition of SHM.

SHM: Acceleration is directly proportional to the displacement from the equilibrium (mean) position and always act in opposite direction to the displacement

Simple Harmonic Motion

Displacement, Velocity and Acceleration as a function of time.

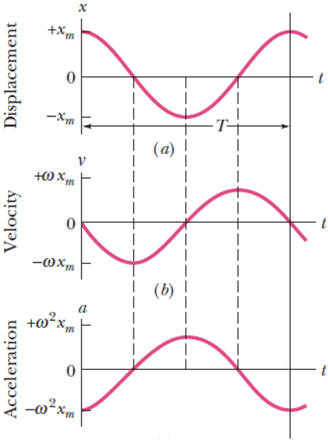


Figure 3: Figure Adopted from HRW

Mass Spring System (A SHM)

Let suppose we have a horizontal, mass spring system. The block is oscillating on a frictional surface.

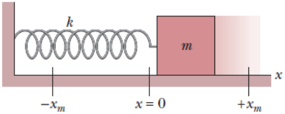


Figure 4: Figure Adopted from HRW

Mass Spring System (A SHM)

The restoring force of the spring is

$$F = -kx$$

From Newton 2nd law of motion

$$F = ma = m(-\omega^2 x)$$

comparing the two equations

$$m\omega^2 = k$$

$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$

This is the angular frequency of a mass spring system.

Mass Spring System (A SHM)

Also

$$\boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$

This is the time period of the mass spring system.
From these two equation we can see that stiff spring will oscillate faster (High frequency).

Example

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

Example

(a) What are the angular frequency, the frequency, and the period of the resulting motion?
Since angular frequency is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{65}{0.68}} = 9.78\ rad/s$$

Frequency is

$$f = \frac{\omega}{2\pi} = \frac{9.78}{2(\frac{22}{7})} = 1.6\ Hz$$

Time period is

$$T = \frac{1}{f} = \frac{1}{1.6} = 0.64\ s$$

Example

(b) What is the amplitude of the oscillation?

$$x_m = 11\ cm$$

(c) What is the maximum speed v_m of the oscillating block, and where is the block when it has this speed?

$$v_m = \omega x_m = (9.78)(0.11) = 1.1\ m/s$$

The block is at the mean position ($x = 0$).

Example

(d) What is the magnitude a_m of the maximum acceleration of the block?

$$a_m = \omega^2 x_m = (9.78)(0.11) = 11\ m/s^2$$

(e) What is the phase constant f for the motion?

From

$$x(t) = x_m cos(\omega t + \phi)$$

$$cos\ \phi = 1$$

$$\phi = 0\ rad$$

Example

(f) What is the displacement function x(t) for the spring–block system?

$$x(t) = 0.11\ cos(9.8\ t)$$

Energy in Simple Harmonic Motion

In Physics-1 we saw that energy of an object in a gravitational field change between Kinetic Energy and Potential Energy and the sum of the two always remains constant. Here we will show the same mechanical energy conservation principle for SHM.

Lets consider the mass spring example again. In Physics-1 we saw that the elastic potential energy stored in a spring is

$$U(t) = \frac{1}{2}kx^2$$

Putting the value of $x(t)$.

$$U(t) = \frac{1}{2}kx_m^2 \cos^2(\omega t + \phi)$$

Energy in Simple Harmonic Motion

Kinetic energy is

$$K(t) = \frac{1}{2}mv^2$$

putting the value of $v(t)$

$$K(t) = \frac{1}{2}m\omega^2 x_m^2 \sin^2(\omega t + \phi)$$

Then

$$K(t) = \frac{1}{2}kx_m^2 \sin^2(\omega t + \phi)$$

Now total mechanical energy is the sum of these two energies

$$E = U + K$$

Energy in Simple Harmonic Motion

$$E = \tfrac{1}{2}kx_m^2\cos^2(\omega t + \phi) + \tfrac{1}{2}kx_m^2\sin^2(\omega t + \phi)$$

$$E = \tfrac{1}{2}kx_m^2\left[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)\right]$$

$$\cos^2\alpha + \sin^2\alpha = 1$$

$$E = U + K = \tfrac{1}{2}kx_m^2$$

Total mechanical energy is

$$\boxed{E = \frac{1}{2}kx_m^2}$$

Which is constant and not changing over time.

Example

Suppose a block has mass $m = 2.72 \times 10^5 kg$ and is designed to oscillate at frequency $f = 10.0 Hz$ and with amplitude $x_m = 20.0 cm$.
(a) What is the total mechanical energy E of the spring-block system?
Since total mechanical energy is

$$E = \frac{1}{2} k x_m^2$$

where

$$k = m\omega^2 = m(2\pi f)^2 = (2.72 \times 10^5)(2\pi)^2(10)^2 = 1.037 \times 10^9 N/m$$

$$E = \frac{1}{2}(1.037 \times 10^9)(20 \times 10^{-2})^2$$

$$\boxed{E = 2.1 \times 10^7 J}$$

Example

(b) What is the block's speed as it passes through the equilibrium point? At equilibrium point the total mechanical energy will be all converted to the kinetic energy.

$$K = E = 2.1 \times 10^7 J$$

$$\frac{1}{2}mv^2 = 2.1 \times 10^7 J$$

$$\boxed{v = 12.6\ m/s}$$