Affine transformations

- In order to incorporate the idea that both the basis and the origin can change, we augment the linear space **u**, **v** with an origin **t**.
- Note that while **u** and **v** are **basis vectors**, the origin **t** is a **point**.
- We call **u**, **v**, and **t** (basis and origin) a **frame** for an **affine space**.
- Then, we can represent a change of frame as:

$$\mathbf{p}' = x \cdot \mathbf{u} + y \cdot \mathbf{v} + \mathbf{t}$$

- This change of frame is also known as an **affine transformation**.
- How do we write an affine transformation with matrices?

Homogeneous Coordinates

- To represent transformations among affine frames, we can loft the problem up into 3-space, adding a third component to every point:

$$
\begin{aligned}
\mathbf{p}' &= \mathbf{M}\mathbf{p} \\
&= \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{t} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\
&= x \cdot \mathbf{u} + y \cdot \mathbf{v} + 1 \cdot \mathbf{t}
\end{aligned}
$$

- Note that $[a\ c\ 0]^{\mathrm{T}}$ and $[b\ d\ 0]^{\mathrm{T}}$ represent vectors and $[t_x\ t_y\ 1]^{\mathrm{T}}$, $[x\ y\ 1]^{\mathrm{T}}$ and $[x'\ y'\ 1]^{\mathrm{T}}$ represent points.

Homogeneous coordinates

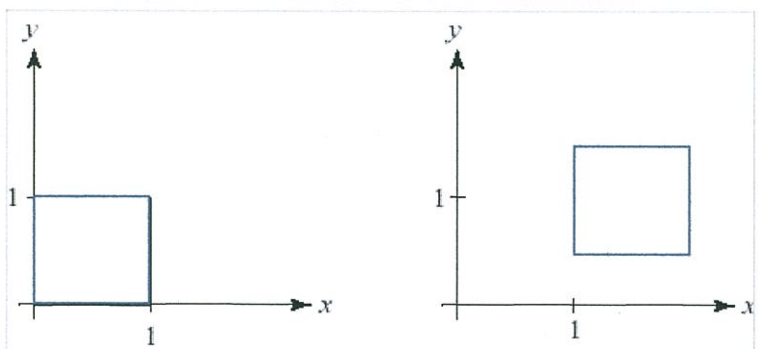
This allows us to perform translation as well as the linear transformations as a matrix operation:

$$\mathbf{p}' = \mathbf{M_T p}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$x' = x + t_x$$

$$y' = y + t_y$$

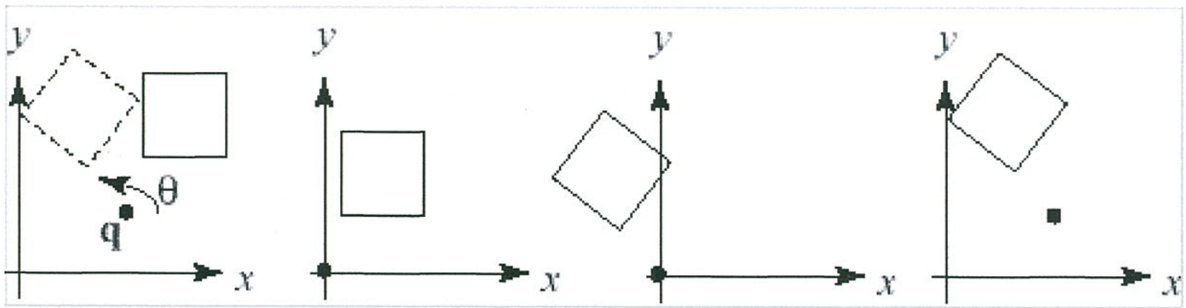


$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation about arbitrary points

Until now, we have only considered rotation about the origin.

With homogeneous coordinates, you can specify a rotation, $\mathbf{R_q}$, about any point $\mathbf{q} = [q_x\ q_y\ 1]^T$ with a matrix:



1. Translate **q** to origin
2. Rotate
3. Translate back

Line up the matrices for these step in right to left order and multiply.

<u>Note</u>: Transformation order is important!!