MAT 3-121 Calculus of a Single Variable

Exam 2 Part 2 November 14, 2016

Solution

name

For credit on these problems, you must show your work. Don't approximate your answers unless directed to do so. Graphing calculators are allowed, but not calculators with symbolic capabilities.

1. (12 pts.) Find the derivative of the following functions:

a. $f(x) = 3^x \tan(2\pi x)$

$$\ln 3 \cdot 3^x \tan(2\pi x) + 3^x \sec^2(2\pi x) \cdot 2\pi$$

b. $h(x) = \sqrt{x}\ln(2 + \sin(3x))$

$$\frac{\ln(2+\sin 3x)}{2\sqrt{x}} + \sqrt{x} \cdot \frac{3\cos 3x}{2+\sin 3x}$$

c. $j(x) = x^{\ln x}$

$y = x^{\ln x}$

$\ln y = \ln x \cdot \ln x = (\ln x)^2$

$(\ln y)' = \frac{2\ln x}{x}$

$y' = x^{\ln x} \cdot \frac{2\ln x}{x}$

d. $k(x) = e^{\pi}$ constant

$k' = 0$

2. (3 pts.) Calculate the anti-derivative of the following function:

$h(x) = \frac{\cos x}{2 + \sin x}$ $\quad \ln(2+\sin x) + C$

3. (3 pts.) Let $h(x) = \frac{f(x)}{g(x)}$. Fill in the missing values in the table:

$$h'(x) = \frac{f'(x) \cdot g(x) - f(x) g'(x)}{(g(x))^2}$$

x	f(x)	f'(x)	g(x)	g'(x)	h'(x)
-2	1	-1	-3	4	-1/9
-1	0	-2	1	1	-2
0	-1	2	-2	1	-3/4
1	2	-2	-1	2	-2
2	3	-1	2	-2	1

$\frac{-2 \cdot 1 - 0 \cdot 1}{1^2}$

$\frac{2 \cdot -2 - -1 \cdot 1}{(-2)^2}$

$\frac{-2 \cdot -1 - 2 \cdot 2}{(-1)^2}$

4. (10 pts.) Consider the implicit function $y^2 + 4x = 4xy^2$

a. Find $\frac{dy}{dx}$

5

$$2yy' + 4 = 4y^2 + 4x \cdot 2yy'$$

$$(2y - 8xy)y' = 4y^2 - 4$$

$$y' = \frac{4y^2 - 4}{2y - 8xy}$$

b. Find the equation of the tangent line to the curve at the point $(\frac{1}{3}, 2)$.

3

$$\frac{4 \cdot 2^2 - 4}{2 \cdot 2 - 8 \cdot \frac{1}{3} \cdot 2} = \frac{12}{4 - \frac{16}{3}} = \frac{12}{-\frac{3}{4}} = -9$$

$$2 = -9 \cdot \frac{1}{3} + b$$

$$b = 5$$

$$y = -9x + 5$$

c. For which values of x and y is the tangent line to this curve vertical?

2

$$2y - 8xy = 0$$

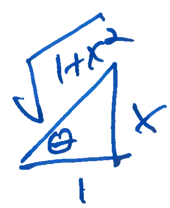
$$2y(1 - 4x) = 0$$

5. (6 pts.) Use implicit differentiation to derive the formula for the derivative of arctan x.

$$\tan(\arctan x) = x$$

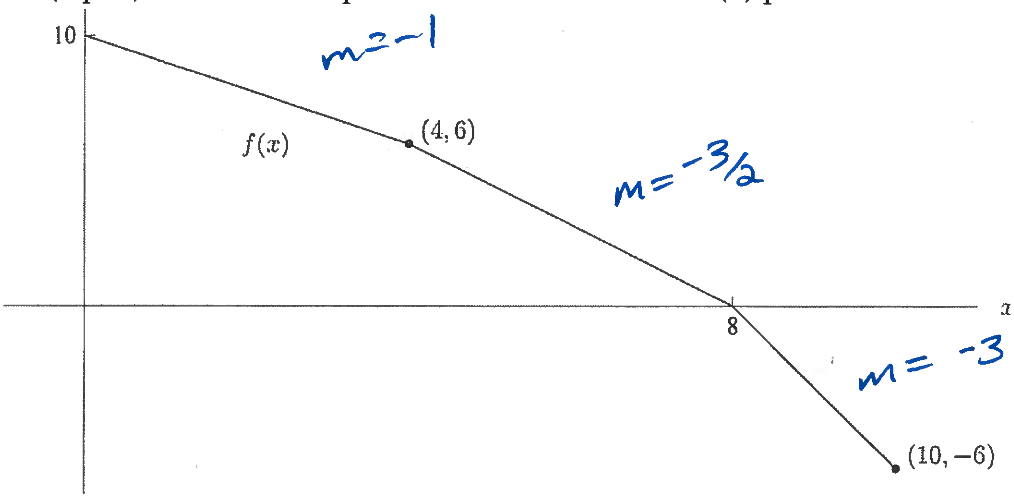
$$\sec^2(\arctan x) \cdot (\arctan x)' = 1$$

$$(\arctan x)' = \frac{1}{\sec^2(\underset{\theta}{\arctan x})}$$



$$= \frac{1}{(\sqrt{1+x^2})^2} = \frac{1}{1+x^2}$$

6. (9 pts.) Consider the piecewise linear function f(x) picture here:



For each function g(x) below, find the value of g′(3). Show your work.

a. $g(x) = \sin([f(x)]^3)$

$$g'(3) = \cos([\overset{7}{f(3)}]^3) \cdot 3[f(3)]^2 \cdot f'(3)$$

$$= \cos 343 \cdot 147 \cdot -1$$

b. $g(x) = \frac{f(x^2)}{x}$

$$\frac{f'(9)\cdot 6\cdot 3 - f(9)}{9} = \frac{-54 - -3}{9} = \frac{-51}{9}$$

c. $g(x) = \ln(f(x)) + \cancel{f(2)} \rightarrow 0$

$$\frac{1}{f(3)} \cdot f'(3) = \frac{1}{7} \cdot -1$$

7. (4 pts.) Does the function $y(x) = 3e^{x^2}$ satisfy the differential equation y′ = 2xy? Justify your answer.

LHS $(3e^{x^2})' = 3e^{x^2} \cdot 2x$

RHS $2x \cdot 3e^{x^2}$ = ✓

8. (12 pts.) Evaluate the following limits or explain why they don't exist. Show your work and justify your answer.

a. $\lim_{x \to \infty} \frac{x^2}{2^x}$ $\boxed{0}$ $\overset{L'H}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x \to \infty} \frac{2x}{\ln 2\, 2^x} \overset{L'H}{\underset{\left(\frac{\infty}{\infty}\right)}{=}} \lim_{x \to \infty} \frac{2}{(\ln 2)^2 \cdot 2^x} = 0$

b. $\lim_{x \to 0} \frac{5x - \sin x}{x}$ $\boxed{4}$ $\overset{L'H}{\underset{\frac{0}{0}}{=}} \lim_{x \to 0} \frac{5 - \cos x}{1} = \frac{5-1}{1} = 4$

c. $\lim_{x \to 0^-} \frac{\cos x}{x}$ $\boxed{-\infty}$ From the left, always negative $\left(\frac{1}{0}\right)$ not indeterminate & getting large.

d. $\lim_{x \to \infty} (1 + \frac{2}{x})^x$

$\lim_{x \to \infty} \ln (1+\frac{2}{x})^x = \lim_{x \to \infty} \frac{\ln(1+\frac{2}{x})}{\frac{1}{x}}$

$= \lim_{x \to \infty} \frac{-\frac{2}{x^2}}{(1+\frac{2}{x}) \cdot \frac{-1}{x^2}} = \lim_{x \to \infty} \frac{2}{1+\frac{2}{x}} = 2$ so e^2

9. (8 pts.) Which point on the line y = -2x + 2 is closest to (0,1). Clearly label your objective function and constraint.

cons: $y = -2x + 2$

obj: $D^2 = x^2 + (y-1)^2$

$f(x) = 5x^2 - 8x + 4$

$f'(x) = 10x - 8$

$f'(x) = 0$ when $x = \frac{8}{10} = \frac{4}{5}$

$y = \frac{2}{5}$

10. (6 pts.) Suppose on the planet Mars, the Mars Rover drops a rock from 1 meter high. This physical system can be modeled by the IVP: y''(t) = -3.7, y(0) = 1, y'(0) = 0. Solve this differential equation to answer the question: How long does the rock take to hit the ground, to the nearest tenth of a second?

$y(t) = -1.85\, t^2 + 1$

$t = \sqrt{\frac{1}{1.85}} \doteq 0.735$

11. (9 pts.) For this problem, give the Differential Equation or Initial Value Problem that models the given physical situation. DO NOT SOLVE.

a. In a model ignoring friction, the change in the velocity of a falling object on Earth is equal to the constant -9.8. Adjust this model to include drag force which acts in the opposite direction at a force proportional to the velocity. Give the differential equation for v(t).

$$v'(t) = -9.8 - kv(t)$$

b. A flu epidemic spreads through a 1100-student college community at a rate proportional to the product of the number of members already infected (represented by I(t)) and the number of those not infected. The epidemic starts with just 5 cases.

$$I'(t) = k \cdot I(t)(1100 - I(t))$$

$$I(0) = 5$$

c. Experience suggests that the rate at which people contribute in a charity drive is proportional to the difference between the current total and the announced goal. A fund drive is announced with a goal of \$90,000 and an initial contribution of \$10,000.

$$c'(t) = k(90{,}000 - c(t))$$

$$c(0) = 10000$$

12. (8 pts.) Produce the slope field (at integer pairs for t from 0 to 1 and y from 0 to 3) for the differential equation y′ = y(3-y). Based on your slope field, what do you expect the long term behavior to be for a solution with initial value y(0) = 1?

It will be increasing towards y=3, then will level off