Specification of Vectors and Matrices

CSC321 Computer Graphics

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1 Vector2D

1.1 Addition

Here is the general rule:

$$\begin{aligned}\vec{u} &= (u_x, u_y)\\ \vec{v} &= (v_x, v_y)\\ \vec{u}+\vec{v} &= (u_x+v_x, u_y+v_y)\end{aligned}$$

Here is a specific example:

$$\begin{aligned}\vec{u} &= (3, 4)\\ \vec{v} &= (5, 12)\\ \vec{u}+\vec{v} &= (3+5, 4+12)\\ &= (8, 16)\end{aligned}$$

1.2 Subtraction

Here is the general rule:

$$\begin{aligned}\vec{u} &= (u_x, u_y)\\ \vec{v} &= (v_x, v_y)\\ \vec{u}+\vec{v} &= (u_x-v_x, u_y-v_y)\end{aligned}$$

Here is a specific example:

$$\begin{aligned}\vec{u} &= (3, 4)\\ \vec{v} &= (5, 12)\\ \vec{v}-\vec{u} &= (5-3, 12-4)\\ &= (2, 8)\end{aligned}$$

1.3 Multiplication by a scalar

Here is the general rule:

$$\begin{aligned} \vec{u} &= (u_x, u_y) \\ scaleFactor \times \vec{u} &= (scaleFactor \times u_x, scaleFactor \times u_y) \end{aligned}$$

Here is a specific example:

$$\begin{aligned} \vec{u} &= (3, 4) \\ 2\vec{u} &= (6, 8) \end{aligned}$$

1.4 Dot product

Here is the general rule:

$$\begin{aligned} \vec{u} &= (u_x, u_y) \\ \vec{v} &= (v_x, v_y) \\ \vec{u} \cdot \vec{v} &= u_x v_x + u_y v_y \end{aligned}$$

Here is a specific example:

$$\begin{aligned} \vec{u} &= (3, 4) \\ \vec{v} &= (5, 12) \\ \vec{u} \cdot \vec{v} &= 3 \cdot 5 + 4 \cdot 12 \\ &= 15 + 48 \\ &= 63 \end{aligned}$$

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \psi \quad \text{where } \psi \text{ is the angle between the vectors}$$

1.5 Magnitude

Here is the general rule:

$$\begin{aligned} \vec{u} &= (u_x, u_y) \\ |\vec{u}| &= \sqrt{u_x^2 + u_y^2} \\ &= \sqrt{\vec{u} \cdot \vec{u}} \end{aligned}$$

Here is a specific example:

$$\begin{aligned}\vec{u} &= (3, 4)\\ |\vec{u}| &= \sqrt{3^2 + 4^2}\\ &= \sqrt{9 + 16}\\ &= \sqrt{25}\\ &= 5\end{aligned}$$

1.6 Normalize

Here is the general rule:

$$\begin{aligned}\vec{u} &= (u_x, u_y)\\ \hat{u} &= \frac{1}{|\vec{u}|}\vec{u}\\ &= (\frac{u_x}{\sqrt{u_x^2 + u_y^2}}, \frac{u_y}{\sqrt{u_x^2 + u_y^2}})\end{aligned}$$

Here is a specific example:

$$\begin{aligned}\vec{u} &= (3, 4)\\ |\vec{u}| &= 5\\ \hat{u} &= \frac{1}{|\vec{u}|}\vec{u}\\ &= (\frac{3}{5}, \frac{4}{5})\end{aligned}$$

2 Vector3D

Here is the general rule:

2.1 Addition

$$\begin{aligned}\vec{u} &= (u_x, u_y, u_z)\\ \vec{v} &= (v_x, v_y, v_z)\\ \vec{u} + \vec{v} &= (u_x + v_x, u_y + v_y, u_z + v_z)\end{aligned}$$

Here is a specific example:

$$\begin{aligned}\vec{u} &= (1, 2, 3)\\ \vec{v} &= (4, 5, 6)\\ \vec{u} + \vec{v} &= (1 + 4, 2 + 5, 3 + 6)\\ &= (5, 7, 9)\end{aligned}$$

2.2 Subtraction

Here is the general rule:

$$
\begin{aligned}
\vec{u} &= (u_x, u_y, u_z) \\
\vec{v} &= (v_x, v_y, v_z) \\
\vec{u} + \vec{v} &= (u_x - v_x, u_y - v_y, u_z - v_z)
\end{aligned}
$$

Here is a specific example:

$$
\begin{aligned}
\vec{u} &= (1, 2, 3) \\
\vec{v} &= (4, 5, 6) \\
\vec{v} - \vec{u} &= (4 - 1, 5 - 2, 6 - 3) \\
&= (3, 3, 3)
\end{aligned}
$$

2.3 Multiplication by a scalar

Here is the general rule:

$$
\begin{aligned}
\vec{u} &= (u_x, u_y) \\
scaleFactor \times \vec{u} &= (scaleFactor \times u_x, scaleFactor \times u_y)
\end{aligned}
$$

Here is a specific example:

$$
\begin{aligned}
\vec{u} &= (1, 2, 3) \\
2\vec{u} &= (2, 4, 6)
\end{aligned}
$$

2.4 Dot product

Here is the general rule:

$$
\begin{aligned}
\vec{u} &= (u_x, u_y, u_z) \\
\vec{v} &= (v_x, v_y, v_z) \\
\vec{u} \cdot \vec{v} &= u_x v_x + u_y v_y + u_z v_z
\end{aligned}
$$

Here is a specific example:

$$
\begin{aligned}
\vec{u} &= (1, 2, 3) \\
\vec{v} &= (4, 5, 6) \\
\vec{u} \cdot \vec{v} &= 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 \\
&= 4 + 10 + 18 \\
&= 30
\end{aligned}
$$

$$
\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \psi \quad \text{where } \psi \text{ is the angle between the vectors}
$$

2.5 Magnitude

Here is the general rule:

$$\begin{aligned}\vec{u} &= (u_x, u_y, u_z)\\ |\vec{u}| &= \sqrt{u_x^2 + u_y^2 + u_z^2}\\ &= \sqrt{\vec{u}\cdot\vec{u}}\end{aligned}$$

Here is a specific example:

$$\begin{aligned}\vec{u} &= (1, 2, 2)\\ |\vec{u}| &= \sqrt{1^2 + 2^2 + 2^2}\\ &= \sqrt{1 + 4 + 4}\\ &= \sqrt{9}\\ &= 3\end{aligned}$$

2.6 Normalize

Here is the general rule:

$$\begin{aligned}\vec{u} &= (u_x, u_y, u_z)\\ \hat{u} &= \frac{1}{|\vec{u}|}\vec{u}\\ &= (\frac{u_x}{\sqrt{u_x^2 + u_y^2 + u_z^2}}, \frac{u_y}{\sqrt{u_x^2 + u_y^2 + u_z^2}})\end{aligned}$$

Here is a specific example:

$$\begin{aligned}\vec{u} &= (1, 2, 2)\\ |\vec{u}| &= 3\\ \hat{u} &= \frac{1}{|\vec{u}|}\vec{u}\\ &= (\frac{1}{3}, \frac{2}{3}, \frac{2}{3})\end{aligned}$$

3 Matrix2x2

3.1 Special 2×2 matrices

3.1.1 Identity

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3.1.2 Rotation

Here is the general rule:

$$R(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}$$

Here is a specific example:

$$R(\pi/2) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

3.1.3 Scaling

$$S(s_x, s_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

Here is a specific example:

$$S(2, 2) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

3.2 Multiplication: matrix × matrix

Here is the general rule:

$$A = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix}$$
$$B = \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix}$$
$$AB = \begin{bmatrix} (a_{00}b_{00} + a_{01}b_{10}) & (a_{00}b_{01} + a_{01}b_{11}) \\ (a_{10}b_{00} + a_{11}b_{10}) & (a_{10}b_{01} + a_{11}b_{11}) \end{bmatrix}$$

Here is a specific example: A rotation by 30° ($\pi/6$ radians) followed by a rotation by 60° ($\pi/3$ radians) produces the same result as a single rotation by 90° ($\pi/2$ radians).

$$R(\frac{\pi}{6}) = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$
$$R(\frac{\pi}{3}) = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$
$$R(\frac{\pi}{2}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$
$$R(\frac{\pi}{6})\ R(\frac{\pi}{3}) = R(\frac{\pi}{2})$$

3.3 Multiplication: matrix × vector

Here is the general rule:

$$
\begin{aligned}
A &= \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \\
\vec{v} &= \begin{bmatrix} v_x \\ v_y) \end{bmatrix} \\
A\vec{v} &= \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} \\
&= \begin{bmatrix} a_{00}v_x + a_{01}v_y \\ a_{10}v_x + a_{11}v_y \end{bmatrix}
\end{aligned}
$$

Here is a specific example:

3.4 Determinant

Here is the general rule:

$$
\begin{aligned}
A &= \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \\
|A| &= a_{00}a_{11} - a_{10}a_{01}
\end{aligned}
$$

Here is a specific example:

$$
\begin{aligned}
A &= \begin{bmatrix} 3 & 2 \\ 6 & 8 \end{bmatrix} \\
|A| &= 3 \cdot 8 - 2 \cdot 6 \\
&= 24 - 12 \\
&= 12
\end{aligned}
$$

3.5 Inverse

Here is the general rule:

$$
\begin{aligned}
A &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
A^{-1} &= \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\
AA^{-1} &= I
\end{aligned}
$$

Here is a specific example:

$$A = \begin{bmatrix} 3 & 2 \\ 6 & 8 \end{bmatrix}$$
$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} 8 & -2 \\ -6 & 3 \end{bmatrix}$$
$$A^{-1} = \frac{1}{12} \begin{bmatrix} 8 & -2 \\ -6 & 3 \end{bmatrix}$$
$$A^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$
$$\begin{bmatrix} 3 & 2 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4 Matrix3x3

4.1 Special 3×3 matrices

4.1.1 Identity

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4.1.2 Rotation about the x-axis

$$R_x(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$

4.1.3 Rotation about the y-axis

$$R_y(\psi) = \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix}$$

4.1.4 Rotation about the z-axis

$$R_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4.1.5 Scaling

$$S(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}$$

4.2 Multiplication: matrix $\times$ matrix

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{bmatrix}$$

$$\begin{aligned} C &= AB \\ &= \begin{bmatrix} (a_{00}b_{00} + a_{01}b_{10} + a_{02}b_{20}) & (a_{00}b_{01} + a_{01}b_{11} + a_{02}b_{21}) & (a_{00}b_{02} + a_{01}b_{12} + a_{02}b_{22}) \\ (a_{10}b_{00} + a_{11}b_{10} + a_{12}b_{20}) & (a_{10}b_{01} + a_{11}b_{11} + a_{12}b_{21}) & (a_{10}b_{02} + a_{11}b_{12} + a_{12}b_{22}) \\ (a_{20}b_{00} + a_{21}b_{10} + a_{22}b_{20}) & (a_{20}b_{01} + a_{21}b_{11} + a_{22}b_{21}) & (a_{20}b_{02} + a_{21}b_{12} + a_{22}b_{22}) \end{bmatrix} \end{aligned}$$

Let c_{ij} be the element in the i^{th} row and j^{th} column of the 3×3 matrix C. Similarly, let a_{ij} and b_{ij} be elements of the 3×3 matrices A and B whose product is C.

Then...

$$\begin{aligned} c_{ij} &= \sum_{k=0}^{2} a_{ik}\ b_{kj} \\ &= a_{i0}\ b_{0j} + a_{i1}\ b_{1j} + a_{i2}\ b_{2j} \end{aligned}$$

4.3 Multiplication: matrix $\times$ vector

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix}$$

$$A\vec{v} = \begin{bmatrix} a_{00}\ v_0 + a_{01}\ v_1 + a_{02}\ v_2 \\ a_{10}\ v_0 + a_{11}\ v_1 + a_{12}\ v_2 \\ a_{20}\ v_0 + a_{21}\ v_1 + a_{22}\ v_2 \end{bmatrix}$$

The element in the i^{th} row of $A\vec{v}$ is $\sum_{k=0}^{2} a_{ik}\ v_k$.