Notes

CSC321 Computer Graphics

12 December 2016

Review before beginning the lesson.

In this lesson, we will multiply matrices and vectors.

Recall that...

$$\begin{bmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} m_{00}\, v_0 + m_{01}\, v_1 \\ m_{10}\, v_0 + m_{11}\, v_1 \end{bmatrix}$$

For example...

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} &= \begin{bmatrix} 1 \cdot 5 + 2 \cdot 6 \\ 3 \cdot 5 + 4 \cdot 6 \end{bmatrix} \\ &= \begin{bmatrix} 17 \\ 39 \end{bmatrix} \end{aligned}$$

Also recall that...

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} = \begin{bmatrix} (a_{00}\, b_{00} + a_{01}\, b_{10}) & (a_{00}\, b_{01} + a_{01}\, b_{11}) \\ (a_{10}\, b_{00} + a_{11}\, b_{10}) & (a_{10}\, b_{01} + a_{11}\, b_{11}) \end{bmatrix}$$

Here is a numerical example...

$$\begin{bmatrix} 2 & 3 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} 2 & -\frac{3}{4} \\ -1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Show that for some matrices A and B it is true that $AB = BA$.
- Show that for some matrices A and B it is not true that $AB = BA$.

Different ways of describing a line in the plane.

Explicit equation.

This explicit equation defines the line in the plane whose slope is m and whose y intercept is b. We can use this form to describe any line except those that are parallel to the y axis.

$$y = m\,x + b$$

- What is the value of m for a horizontal line?

Parametric equations.

This is the parametric formulation of the line that passes through the points (x_0, y_0) and (x_1, y_1). We can use this form to describe any line (including lines that are parallel to the y axis).

$$x(t) = (1-t)\;x_0 + t\;x_1$$
$$y(t) = (1-t)\;y_0 + t\;y_1$$

When $0 \le t \le 1$, $(x(t), y(t))$ is a point on the line segment whose endpoints are (x_0, y_0) and (x_1, y_1).

When $t < 0$ or $t > 1$, the point $(x(t), y(t))$ lies on the line that passes through (x_0, y_0) and (x_1, y_1) but it is outside the segment bounded by the two endpoints.

- Describe a line parametrically with a single vector equation.

Implicit equation.

We can also describe any line with an implicit equation.

$$a\,x + b\,y + c = 0$$

Of course, the equality holds when we multiply both sides by the same value and when we subtract the same value from both sides.

$$\begin{aligned} a\,x + b\,y + c &= 0 \\ a\,x + b\,y &= -c \\ \frac{-1}{c}\,(a\,x + b\,y) &= -c \\ \frac{-a}{c}\,x + \frac{-b}{c}\,y &= 1 \end{aligned}$$

By replacing the values of a, b, and c we can describe any line with an equation that looks like this:

$$a\,x + b\,y = 1$$

The equation holds for all points (x, y) on the line.

- Let $f(x, y) = a\,x + b\,y + c$ and let $f(x, y) = 0$ be the implicit equation of a line.

  Find the elements of a vector that points along the direction of the line by computing the gradient of $f(x, y)$:

  Do this by first differentiating $f(x, y)$ with respect to x, treating y as a constant. Then differentiate $f(x, y)$ with respect to y, treating x as a constant.

$$\nabla f(x, y) = \left(\frac{\partial f(x, y)}{\partial x}, \frac{\partial f(x, y)}{\partial y} \right)$$

Finding the implicit equation of a line.

Given two distinct points on the line, we can find the values of a and b by solving a system of linear equations.

$$\begin{aligned} a\,x_0 + b\,y_0 &= 1 \\ a\,x_1 + b\,y_1 &= 1 \end{aligned}$$

We can describe this relationship with a matrix of coefficients, a vector of the unknowns, and a vector of constants on the right-hand side.

$$\begin{bmatrix} x_0 & y_0 \\ x_1 & y_1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This is a system of two equations with two unknowns.

We can solve the system by multiplying both sides by the inverse of the matrix of coefficients.

If A is a 2×2 matrix, then its determinant $\mid A \mid$ and inverse A^{-1} are given as follows:

$$\begin{aligned} A &= \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \\ \mid A \mid &= a_{00}\,a_{11} - a_{01}\,a_{10} \\ A^{-1} &= \frac{1}{\mid A \mid} \begin{bmatrix} a_{11} & -a_{01} \\ -a_{10} & a_{00} \end{bmatrix} \end{aligned}$$

We can confirm that A^{-1} is the inverse of A by multiplying the two matrices:

$$A^{-1}\,A = I$$

$$\frac{1}{a_{00}\,a_{11} - a_{01}\,a_{10}} \begin{bmatrix} a_{11} & -a_{01} \\ -a_{10} & a_{00} \end{bmatrix} \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, let's use our knowledge of inverses to find the implicit equation of a line that passes through (x_0, y_0) and (x_1, y_1).

$$\begin{bmatrix} x_0 & y_0 \\ x_1 & y_1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{1}{x_0\, y_1 - y_0\, x_1} \begin{bmatrix} y_1 & -y_0 \\ -x_1 & x_0 \end{bmatrix} \begin{bmatrix} x_0 & y_0 \\ x_1 & y_1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{x_0\, y_1 - y_0\, x_1} \begin{bmatrix} y_1 & -y_0 \\ -x_1 & x_0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{x_0\, y_1 - y_0\, x_1} \begin{bmatrix} y_1 & -y_0 \\ -x_1 & x_0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{x_0\, y_1 - y_0\, x_1} \begin{bmatrix} (y_1 - y_0) \\ (x_0 - x_1) \end{bmatrix}$$

- When is it not possible to solve a system of two linear equations with two unknowns?

The implicit equation of the line that passes through the points (x_0, y_0) and (x_1, y_1) is:

$$\frac{y_1 - y_0}{x_0\, y_1 - y_0\, x_1}\, x + \frac{x_0 - x_1}{x_0\, y_1 - y_0\, x_1}\, y = 1$$

$$(y_1 - y_0)\, x + (x_0 - x_1)\, y = (x_0\, y_1 - y_0\, x_1)$$

$$(y_1 - y_0)\, x + (x_0 - x_1)\, y + (y_0\, x_1 - x_0\, y_1) = 0$$

This equation holds for x and y when the point (x, y) lies on the line that passes through the points (x_0, y_0) and (x_1, y_1).

- Confirm that the equality holds for $x = x_0$ and $y = y_0$.
- Confirm that the equality holds for $x = x_1$ and $y = y_1$.

For points not on the line, evaluation of the left-hand side of this equation yields a value other than 0. That non-zero value represents a "signed distance." A point whose distance from the line is negative is on one side of the line. A point whose distance is positive is on the other side of the line.

This formula gives us a mean to distinguish between points that lie on different sides of a line.

The distance is also a scaled distance. We get to choose the scale. The choice of a particular scale will give us another way of specifying the coordinates of a point and a means to determine whether a point is inside or outside of a triangle.

Barycentric coordinates.

Any line in the plane can be described with an implicit equation:

$$a\,x + b\,y + c = 0$$

We have shown how to find values for a, b, and c to describe the line that passes through two points (x_0, y_0) and (x_1, y_1).

$$\begin{aligned} a &= (y_1 - y_0) \\ b &= (x_0 - x_1) \\ c &= (y_0\,x_1 - x_0\,y_1) \end{aligned}$$

$$(y_1 - y_0)\,x + (x_0 - x_1)\,y + (y_0\,x_1 - x_0\,y_1) = 0$$

Of course, if the equation $a\,x + b\,y + c = 0$ holds, so do all of these equations...

$$\begin{aligned} &2\,a\,x + 2\,b\,y + 2\,c = 0 \\ &3\,a\,x + 3\,b\,y + 3\,c = 0 \\ &s\,a\,x + s\,b\,y + s\,c = 0 \quad \text{for all } s \end{aligned}$$

Let us scale our implicit equation for the line that passes through (x_0, y_0) and (x_1, y_1) so that the distance to a third point (x_2, y_2) is 1.0.

$$\frac{(y_1 - y_0)\, x + (x_0 - x_1)\, y + (y_0\, x_1 - x_0\, y_1)}{(y_1 - y_0)\, x_2 + (x_0 - x_1)\, y_2 + (y_0\, x_1 - x_0\, y_1)} = 0$$

- Confirm that the value of the left-side of this equation is 1.0 when $x = x_2$ and $y = y_2$.
- Let (x_0, y_0), (x_1, y_1), and (x_2, y_2) be the three vertices of a triangle. What are the upper and lower bounds of the signed, scaled distances of points (x, y) in the triangle from the edge defined by (x_0, y_0) and (x_1, y_1)?

Distances of a point from the 3 edges of a triangle.

Let (x, y) be a point in the plane.

Let (x_0, y_0), (x_1, y_1), and (x_2, y_2) be the three vertices of a triangle in the plane.

Then the distances of (x, y) from the lines that contain the triangle's edges are given as follows:

$$\alpha = \frac{(y_1 - y_0)\, x + (x_0 - x_1)\, y + (y_0\, x_1 - x_0\, y_1)}{(y_1 - y_0)\, x_2 + (x_0 - x_1)\, y_2 + (y_0\, x_1 - x_0\, y_1)}$$

$$\beta = \frac{(y_2 - y_1)\, x + (x_1 - x_2)\, y + (y_1\, x_2 - x_1\, y_2)}{(y_2 - y_1)\, x_0 + (x_1 - x_2)\, y_0 + (y_1\, x_2 - x_1\, y_2)}$$

$$\gamma = \frac{(y_0 - y_2)\, x + (x_2 - x_0)\, y + (y_2\, x_0 - x_2\, y_0)}{(y_0 - y_2)\, x_1 + (x_2 - x_0)\, y_1 + (y_2\, x_0 - x_2\, y_0)}$$

$$= 1 - \alpha - \beta$$

A point is in the triangle if and only if...

$$0 \leq \alpha \leq 1$$
$$0 \leq \beta \leq 1$$
$$0 \leq \gamma \leq 1$$

- Write a JavaScript function that returns **true** if a point is inside a given triangle and **false** otherwise. Define the function with arguments that specify the triangle and point.