An object moves along an east–west axis; distance is measured in meters and time in seconds. Let $p(t)$, $v(t)$, and $a(t)$ denote, respectively, the object's position (in meters east of the starting point), eastward velocity (in meters per second), and eastward acceleration (in meters per second per second) at time t. Then $p' = v$ and $p'' = v' = a$. In Exercises 32–35, translate the statement into a first-order DE.

32. The object accelerates westward at a constant 5 meters per second per second.

33. The object travels eastward at a constant speed of 15 meters per second.

34. Because of friction, the object's eastward acceleration is proportional to its velocity. [NOTE: The proportionality constant is negative; explain why.]

35. The object's eastward velocity is proportional to the square root of the object's position.

In Exercises 36–38, let $h(t)$ represent the height, in meters above ground level, of an object at time t seconds, and let $v(t) = h'(t)$ represent the object's vertical velocity, in meters per second, at time t seconds.

36. **(a)** Solve the IVP $v'(t) = 1$; $v(0) = 0$.

(b) Assuming the initial condition given in part (a), find the object's upward velocity at $t = 10$ seconds.

37. Find and interpret $h(10)$ if $h'(t) = -3$ and $h(0) = 100$.

38. **(a)** Verify by differentiation that for any constant C, $v(t) = 100/(t + C)$ is a solution of the DE $v' = -0.01v^2$.

(b) Suppose that $v' = -0.01v^2$ and $v(0) = 5$. What is $v(30)$? What is $v(80)$? Interpret your answers in physical language.

39. An object moves along a straight line in such a way that it is slowing down with an acceleration of -8 meters per second per second. The object has an initial velocity of 14 meters per second. Find the distance the object travels during the first 3 seconds.

40. Suppose that the brakes are applied on a car traveling 50 mph and that this gives the car a constant negative acceleration of 20 ft/sec^2. How long will it take the car to come to a stop? How far will the car travel before stopping? [HINT: 1 mph $= 22/15$ ft/sec.]

41. First National Bank bank advertises 8% interest compounded continuously. Second National Bank advertises 10% interest compounded continuously but charges depositors a \$100 yearly administrative fee for the privilege of banking there. (For simplicity, assume that the \$100 fee is deducted continously over the full year.) Let $P(t)$ denote the value in dollars of a deposit after t years.

(a) Explain why First National Bank's policy is modeled by the DE $P'(t) = 0.08P(t)$.

(b) Explain why Second National Bank's policy is modeled by the DE $P' = 0.1P - 100$.

42. A flu epidemic spreads through a 3000-student college community at a rate proportional to the product of the number of members already infected and the number of those not yet infected. (This product measures the number of possible infectious contacts.) Let $P(t)$ represent the number of students infected after t days. Write a differential equation that relates P and P'.

43. The rate at which the temperature of an object changes is proportional to the difference between the temperature T of the object and the temperature S of the object's surroundings. Express this physical law as a differential equation.

44. Oil is being pumped continuously from a well at a rate proportional to the amount of oil left in the well. Write a differential equation satisfied by the function $O(t)$, the amount of oil in the tank at time t.

45. In learning theory, a "performance function" $P(t)$ may be used to measure someone's skill at a task (using units appropriate to the task) as a function of the training time t. The graph of a performance function is called a **learning curve.** For many tasks, performance improves quickly at first but tapers off (i.e., the rate of learning decreases) as the value of $P(t)$ approaches M, some maximal level of performance. Explain how the DE $P' = k(M - P)$, where k is a positive constant, describes this situation.

FURTHER EXERCISES

46. A car traveling at a constant speed of 80 mph along a straight highway fails to stop at a stop sign. Three seconds later, a highway patrol car starts from a point of rest at the stop sign and maintains a constant acceleration of 8 ft/sec^2. How long will it take the patrol car to overtake the speeding automobile? How far from the stop sign will this occur? What is the speed of the patrol car when it overtakes the automobile?

47. Suppose that a number of rabbits are introduced onto an island on which they have no natural enemies but that can support a maximum population of 1000 rabbits. Let $P(t)$ denote the number of rabbits at time t (measured in months) and suppose that P satisfies the differential equation

$$\frac{dP}{dt} = kP(1000 - P),$$

where k is a positive constant.

(a) Suppose that 1000 rabbits are introduced onto the island at time $t = 0$. Does the model predict that the rabbit population will increase, decrease, or remain constant? Justify your answer.

(b) Suppose that 1500 rabbits are introduced onto the island at time $t = 0$. Does the model predict that the rabbit population will increase, decrease, or remain constant? Justify your answer.

12. Find the unique solution of the IVP $y' = 0.06y$, $y(0) = 100$.

13. (a) Solve the IVP $y' = 3y$, $y(0) = 37$.

(b) How many solutions are there?

14. Is $y(t) = 100 + 50e^{2t}$ a solution of the IVP $y' = 2(y - 100)$, $y(0) = 150$? Justify your answer.

In Exercises 15–20, find an expression for f'. Check your answers for reasonableness by plotting the expressions for f and f' on the same axes.

15. $f(x) = 2e^x + \pi$

16. $f(x) = e^x + x^e + e$

17. $f(x) = 2^x + x^2 + 2$

18. $f(x) = 2 \cdot 3^{x-1}$

19. $f(x) = -2 \ln x$

20. $f(x) = 3 \log_2 x$

21–26. Find the second derivative of each function in Exercises 15–20. Check your answers for reasonableness by plotting the expressions for f and f'' on the same axes.

27–30. Find an antiderivative of each function in Exercises 15–18.

31. Find an equation of the line tangent to the curve $y = e^x$ at $x = 0$.

32. Find an equation of the line tangent to the curve $y = \ln x$ at $x = 1$.

33. What is the slope of the curve $y = 3^x$ at $x = 0$?

34. What is the slope of the curve $y = \log_3 x$ at $x = 1$?

35. On January 1, 1960, the population of Boomtown was 50,000. Since then the size of the population has been accurately modeled by the function $P(t) = 50{,}000(0.98)^t$, where t is the number of years since January 1, 1960.

(a) What was the population of Boomtown on January 1, 2000?

(b) At what rate was the population of Boomtown changing on January 1, 2000? Was it increasing or decreasing then?

36. How long does it take for one-quarter of the carbon-14 atoms in a sample to decay?

In Exercises 37 and 38, the position of the particle on the x-axis at time $t > 0$ seconds is $x(t) = \ln t$ meters.

37. (a) Find the average velocity of the particle over the interval $1 \le t \le e$.

(b) Find the instantaneous velocity of the particle at time $t = e$.

38. Find the instantaneous acceleration of the particle at time $t = 2$.

In Exercises 39 and 40, $f(x) = \sqrt{x} - \ln x$.

39. Show that f achieves its (global) minimum value at $x = 4$.

40. Show that f has an inflection point at $x = 16$.

In Exercises 41 and 42, $f(x) = \ln x - e^{x-1}$.

41. Show that f achieves its (global) maximum value at $x = 1$. [Hint: $e^{x-1} = e^x/e$.]

42. Show that f is concave down everywhere on its domain.

In Exercises 43–46, use Theorem 14 to solve the IVP.

43. $y' = 0.1y$; $y(0) = 100$

44. $y' = -0.0001y$; $y(0) = 1$

45. $y' = (\ln 2) \cdot y$; $y(2) = 4$

46. $y' = ky$; $y(10) = 2y(0)$
[Hint: Solve for k using the initial condition.]

47. First National Bank bank advertises 8% interest, compounded continuously. Thus, if $P(t)$ denotes the value in dollars of a deposit after t years, then $P' = 0.08P$. If \$2000 is deposited now in First National Bank, how much will it be worth in 10 years?

Further Exercises

48. Let $f(x) = e^{-x} = \exp(-x)$. Use the fact that $e^{-x} = (1/e)^x$ to show that $f'(x) = -e^{-x}$.

49. Let $f(x) = e^x$.

(a) Find values of the constants a and b so that the linear function $L(x) = a + bx$ has the properties $L(0) = f(0)$

(d) Plot f, L, Q, and C on the same axes over the interval $[-3, 3]$.

(e) Find an interval over which $|f(x) - L(x)| < 0.1$.

(f) Find an interval over which $|f(x) - Q(x)| < 0.1$.

(g) Find an interval over which $|f(x) - C(x)| < 0.1$.

the force on an object. Thus the DE $y'' = -ky$ describes an object that is influenced by a **restoring force** that is *negatively* proportional to the object's displacement y.

PROBLEM 1 Use the chain rule to check that, as claimed above, every function of the form $y = a\sin(\sqrt{k}\,t) + b\cos(\sqrt{k}\,t)$ satisfies the DE $y'' = -ky$. What values of k, a, and b correspond to the functions $y = \cos(3\pi t)$ and $y = 2\cos(4\pi t)$ shown above?

PROBLEM 2 Consider the DE $y'' = ky$, where k is a positive constant. Note that the trigonometric functions just discussed do *not* satisfy the DE. But consider the **hyperbolic functions**

$$\sinh t = \frac{e^t - e^{-t}}{2} \qquad \text{and} \qquad \cosh t = \frac{e^t + e^{-t}}{2}.$$

Show that every function of the form $y = a\sinh(\sqrt{k}\,t) + b\cosh(\sqrt{k}\,t)$ satisfies the DE $y'' = ky$. Plot several solutions to $y'' = y''$ for $0 \le t \le 10$. Interpret the DE and the appearance of solutions in terms of displacement and acceleration.

Damping with DEs Damped vibration, like simple harmonic motion, is linked to a second-order DE but this time of the form

$$y'' = -ky - cy',$$

where k and c are positive constants. As before, $-ky$ represents the restoring force, while the second summand $-cy$ represents a **damping force** that is negatively proportional to the object's *velocity* y'; here c is called the **damping constant.** ➜

Air drag and other frictional forces are often negatively proportional to velocity.

Solutions of the simpler DE $y'' = -ky$ were relatively easy to guess and then check. The present DE is a little more complicated. Still, the following Fact holds:

> **FACT** Consider the DE $y'' = -ky - cy'$, for k and c positive constants. For any constants a and b, the function
>
> $$y = e^{-ct/2}\left(a\cos(\sqrt{k - c^2/4}\,t) + b\sin(\sqrt{k - c^2/4}\,t)\right)$$
>
> is a solution.

Checking that the Fact holds is a matter of a straightforward (but delicate) calculation.

PROBLEM 3 Show by direct calculation that $y = e^{-ct/2}\cos(\sqrt{k - c^2/4}\,t)$ is a solution of $y'' = -ky + cy'$. (A similar calculation (one is enough!) shows that $y = e^{-ct/2}\sin(\sqrt{k - c^2/4}\,t)$ is *also* a solution.)

PROBLEM 4 Suppose that both $y = f(t)$ and $y = g(t)$ are solutions to $y'' = -ky - cy'$; let a and b be constant. Show that every function of the form $y = af(t) + bg(t)$ is also a solution. (This and the preceding problem show that the Fact holds.)

PROBLEM 5 Consider the functions $y = e^{-0.5t}\cos(3\pi t)$ and $y = 3e^{-0.5t}\cos(4\pi t)$, shown graphically above. Find the corresponding values of a, k, and c.

How might one guess the solutions in the preceding Fact? One approach is to look for solutions of the general form $y = e^{-At}\cos(Bt)$ and hope to find appropriate values for A and B. The next problem gives details.

(a) One of the "curves" is a straight line. Which straight line? Label this curve with the appropriate value of C.

(b) One of the curves passes through the point $(0, 4)$. What value of C corresponds to this curve? How do you know? Label this curve with the appropriate value of C.

(c) Estimate (using the graph) the slope at $(0, 4)$ of the curve mentioned in part (b). Does your answer agree with what the DE predicts?

(d) Draw the line $y + t = 0$ (also known as $y = -t$) on the axes. This line crosses four of the solution curves at what points of special interest? Why does this occur? [HINT: What does the DE $y' = y + t$ say about points on the line $y + t = 0$?]

(e) Draw the line $y + t = -3$ (also known as $y = -t - 3$) on the axes. This line crosses four of the solution curves. What do these crossing points have in common? Explain your answer.

(f) Draw several lines with slope -1 on the axes. Each such line crosses several solution curves. What can be said about the points at which these crossings occur?

(g) The curves shown correspond to the C-values $C = \pm 500$, $C = \pm 50$, $C = \pm 5$, and $C = \pm 0.2$. Label each curve with its appropriate C-value.

3. Suppose that $f(t)$ is a solution of the DE $y' = t^2 y + t$ and that $f(1) = 2$. Explain why $y = 3t - 1$ is an equation of the line tangent to f at the point $(1, 2)$.

4. Suppose that $g(t)$ is a solution of the DE $y' = y + 2e^{-t}$ and that $g(0) = 1$. Explain why $y = 3t + 1$ is an equation of the line tangent to g at the point $(0, 1)$.

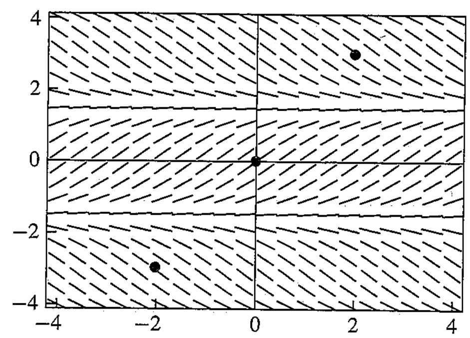
5. Suppose that $h(t)$ is a solution of the DE $y' = t^2 y$ and that $h(-2) = 1$. Find an equation of the line tangent to h at the point $(-2, 1)$.

6. Suppose that $f(t)$ is a solution of the DE $y' = e^t y$ and that $f(0) = -2$. Is $y = t - 2$ an equation of the line tangent to f at the point $(0, -2)$? Justify your answer.

In Exercises 7–14, first match the slope field to one of the differential equations (i)–(x) below. Then draw solution curves through each of the marked points on a copy of the slope field.

(i) $y' = 2t$
(ii) $y' = -2t$
(iii) $y' = ty$
(iv) $y' = -ty$
(v) $y' = y/t$
(vi) $y' = -y/t$
(vii) $y' = \cos y$
(viii) $y' = \sin t$
(ix) $y' = 1 - y$
(x) $y' = y(1 - y)$

7.



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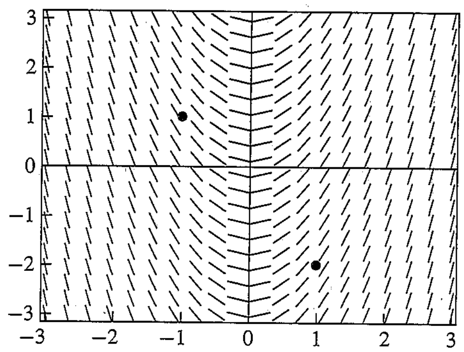

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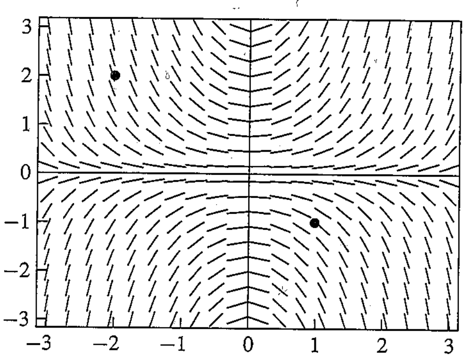

10.