1. Note that $dy/dt = 0$ if and only if $y = -3$. Therefore, the constant function $y(t) = -3$ for all t is the only equilibrium solution.

3. **(a)** The equilibrium solutions correspond to the values of P for which $dP/dt = 0$ for all t. For this equation, $dP/dt = 0$ for all t if $P = 0$ or $P = 230$.

(b) The population is increasing if $dP/dt > 0$. That is, $P(1 - P/230) > 0$. Hence, $0 < P < 230$.

(c) The population is decreasing if $dP/dt < 0$. That is, $P(1 - P/230) < 0$. Hence, $P > 230$ or $P < 0$. Since this is a population model, $P < 0$ might be considered "nonphysical."

4. **(a)** The equilibrium solutions correspond to the values of P for which $dP/dt = 0$ for all t. For this equation, $dP/dt = 0$ for all t if $P = 0$, $P = 50$, or $P = 200$.

(b) The population is increasing if $dP/dt > 0$. That is, $P < 0$ or $50 < P < 200$. Note, $P < 0$ might be considered "nonphysical" for a population model.

(c) The population is decreasing if $dP/dt < 0$. That is, $0 < P < 50$ or $P > 200$.

5. In order to answer the question, we first need to analyze the sign of the polynomial $y^3 - y^2 - 12y$. Factoring, we obtain

$$y^3 - y^2 - 12y = y(y^2 - y - 12) = y(y - 4)(y + 3).$$

(a) The equilibrium solutions correspond to the values of y for which $dy/dt = 0$ for all t. For this equation, $dy/dt = 0$ for all t if $y = -3$, $y = 0$, or $y = 4$.

(b) The solution $y(t)$ is increasing if $dy/dt > 0$. That is, $-3 < y < 0$ or $y > 4$.

(c) The solution $y(t)$ is decreasing if $dy/dt < 0$. That is, $y < -3$ or $0 < y < 4$.

7. The general solution of the differential equation $dr/dt = -\lambda r$ is $r(t) = r_0 e^{-\lambda t}$ where $r(0) = r_0$ is the initial amount.

(a) We have $r(t) = r_0 e^{-\lambda t}$ and $r(5230) = r_0/2$. Thus

$$\frac{r_0}{2} = r_0 e^{-\lambda \cdot 5230}$$

$$\frac{1}{2} = e^{-\lambda \cdot 5230}$$

$$\ln \frac{1}{2} = -\lambda \cdot 5230$$

$$-\ln 2 = -\lambda \cdot 5230$$

because $\ln 1/2 = -\ln 2$. Thus,

$$\lambda = \frac{\ln 2}{5230} \approx 0.000132533.$$

(b) We have $r(t) = r_0 e^{-\lambda t}$ and $r(8) = r_0/2$. By a computation similar to the one in part (a), we have

$$\lambda = \frac{\ln 2}{8} \approx 0.0866434.$$

(c) If $r(t)$ is the number of atoms of C-14, then the units for dr/dt is number of atoms per year. Since $dr/dt = -\lambda r$, λ is "per year." Similarly, for I-131, λ is "per day." The unit of measurement of r does not matter.

(d) We get the same answer because the original quantity, r_0, cancels from each side of the equation. We are only concerned with the proportion remaining (one-half of the original amount).

8. We will solve for k percent. In other words, we want to find t such that $r(t) = (k/100)r_0$, and we know that $r(t) = r_0 e^{-\lambda t}$, where $\lambda = (\ln 2)/5230$ from Exercise 7. Thus we have

$$\begin{aligned} r_0 e^{-\lambda t} &= \frac{k}{100} r_0 \\ e^{-\lambda t} &= \frac{k}{100} \\ -\lambda t &= \ln\left(\frac{k}{100}\right) \\ t &= \frac{-\ln\left(\frac{k}{100}\right)}{\lambda} \\ t &= \frac{\ln 100 - \ln k}{\lambda} \\ t &= \frac{5230(\ln 100 - \ln k)}{\ln 2}. \end{aligned}$$

Thus, there is 88% left when $t \approx 964.54$ years; there is 12% left when $t \approx 15{,}998$ years; 2% left when $t \approx 29{,}517$ years; and 98% left when $t \approx 152.44$ years.

11. The solution of $dR/dt = kR$ with $R(0) = 4{,}000$ is

$$R(t) = 4{,}000\, e^{kt}.$$

Setting $t = 6$, we have $R(6) = 4{,}000\, e^{(k)(6)} = 130{,}000$. Solving for k, we obtain

$$k = \tfrac{1}{6} \ln\left(\tfrac{130{,}000}{4{,}000}\right) \approx 0.58.$$

Therefore, the rabbit population in the year 2010 would be $R(10) = 4{,}000\, e^{(0.58 \cdot 10)} \approx 1{,}321{,}198$ rabbits.

12. **(a)** In this analysis, we consider only the case where v is positive. The right-hand side of the differential equation is a quadratic in v, and it is zero if $v = \sqrt{mg/k}$. Consequently, the solution $v(t) = \sqrt{mg/k}$ for all t is an equilibrium solution. If $0 \le v < \sqrt{mg/k}$, then $dv/dt > 0$, and consequently, $v(t)$ is an increasing function. If $v > \sqrt{mg/k}$, then $dv/dt < 0$, and $v(t)$ is a decreasing function. In either case, $v(t) \to \sqrt{mg/k}$ as $t \to \infty$.

(b) See part (a).

13. The rate of learning is dL/dt. Thus, we want to know the values of L between 0 and 1 for which dL/dt is a maximum. As $k > 0$ and $dL/dt = k(1 - L)$, dL/dt attains it maximum value at $L = 0$.

14. **(a)** Let $L_1(t)$ be the solution of the model with $L_1(0) = 1/2$ (the student who starts out knowing one-half of the list) and $L_2(t)$ be the solution of the model with $L_2(0) = 0$ (the student who starts out knowing none of the list). At time $t = 0$,

$$\frac{dL_1}{dt} = 2\,(1 - L_1(0)) = 2\left(1 - \tfrac{1}{2}\right) = 1,$$

and

$$\frac{dL_2}{dt} = 2\,(1 - L_2(0)) = 2.$$

Hence, the student who starts out knowing none of the list learns faster at time $t = 0$.

(b) The solution $L_2(t)$ with $L_2(0) = 0$ will learn one-half the list in some amount of time $t_* > 0$. For $t > t_*$, $L_2(t)$ will increase at exactly the same rate that $L_1(t)$ increases for $t > 0$. In other words, $L_2(t)$ increases at the same rate as $L_1(t)$ at t_* time units later. Hence, $L_2(t)$ will never catch up to $L_1(t)$ (although they both approach 1 as t increases). In other words, after a very long time $L_2(t) \approx L_1(t)$, but $L_2(t) < L_1(t)$.

15. **(a)** We have $L_B(0) = L_A(0) = 0$. So Aly's rate of learning at $t = 0$ is dL_A/dt evaluated at $t = 0$. At $t = 0$, we have

$$\frac{dL_A}{dt} = 2(1 - L_A) = 2.$$

Beth's rate of learning at $t = 0$ is

$$\frac{dL_B}{dt} = 3(1 - L_B)^2 = 3.$$

Hence Beth's rate is larger.

(b) In this case, $L_B(0) = L_A(0) = 1/2$. So Aly's rate of learning at $t = 0$ is

$$\frac{dL_A}{dt} = 2(1 - L_A) = 1$$

because $L_A = 1/2$ at $t = 0$. Beth's rate of learning at $t = 0$ is

$$\frac{dL_B}{dt} = 3(1 - L_B)^2 = \frac{3}{4}$$

because $L_B = 1/2$ at $t = 0$. Hence Aly's rate is larger.

(c) In this case, $L_B(0) = L_A(0) = 1/3$. So Aly's rate of learning at $t = 0$ is

$$\frac{dL_A}{dt} = 2(1 - L_A) = \frac{4}{3}.$$

Beth's rate of learning at $t = 0$ is

$$\frac{dL_B}{dt} = 3(1 - L_B)^2 = \frac{4}{3}.$$

They are both learning at the same rate when $t = 0$.

19. Several different models are possible. Let $R(t)$ denote the rhinoceros population at time t. The basic assumption is that there is a minimum threshold that the population must exceed if it is to survive. In terms of the differential equation, this assumption means that dR/dt must be negative if R is close to zero. Three models that satisfy this assumption are:

- If k is a growth-rate parameter and M is a parameter measuring when the population is "too small", then
$$\frac{dR}{dt} = kR\left(\frac{R}{M} - 1\right).$$
- If k is a growth-rate parameter and b is a parameter that determines the level the population will start to decrease ($R < b/k$), then
$$\frac{dR}{dt} = kR - b.$$
- If k is a growth-rate parameter and b is a parameter that determines the extinction threshold, then
$$\frac{dR}{dt} = kR - \frac{b}{R}.$$

In each case, if R is below a certain threshold, dR/dt is negative. Thus, the rhinos will eventually die out. The choice of which model to use depends on other assumptions. There are other equations that are also consistent with the basic assumption.

20. **(a)** The relative growth rate for the year 1990 is

$$\frac{1}{s(t)}\frac{ds}{dt} = \frac{1}{5.3}\left(\frac{7.6 - 3.5}{1991 - 1989}\right) \approx 0.387.$$

Hence, the relative growth rate for the year 1990 is 38.7%.

(b) If the quantity $s(t)$ grows exponentially, then we can model it as $s(t) = s_0 e^{kt}$, where s_0 and k are constants. Calculating the relative growth rate, we have

$$\frac{1}{s(t)}\frac{ds}{dt} = \frac{1}{s_0 e^{kt}}\left(k s_0 e^{kt}\right) = k.$$

Therefore, if a quantity grows exponentially, its relative growth rate is constant for all t.

(c)

Year	Rel. Growth Rate	Year	Rel. Growth Rate	Year	Rel. Growth Rate
1991	0.38	1997	0.23	2003	0.13
1992	0.38	1998	0.22	2004	0.13
1993	0.41	1999	0.24	2005	0.12
1994	0.38	2000	0.19	2006	0.09
1995	0.29	2001	0.12	2007	0.06
1996	0.24	2002	0.11		

(d) As shown in part (b), the number of subscriptions will grow exponentially if the relative growth rates are constant over time. The relative growth rates are (roughly) constant from 1991 to 1994, after which they drop off significantly.

(e) If a quantity $s(t)$ grows according to a logistic model, then

$$\frac{ds}{dt} = ks\left(1 - \frac{s}{N}\right),$$

so the relative growth rate

$$\frac{1}{s}\frac{ds}{dt} = k\left(1 - \frac{s}{N}\right).$$

The right-hand side is linear in s. In other words, if s is plotted on the horizontal axis and the relative growth rate is plotted on the vertical axis, we obtain a line. This line goes through the points $(0, k)$ and $(N, 0)$.

21. **(a)** The term governing the effect of the interaction of x and y on the rate of change of x is $+\beta xy$. Since this term is positive, the presence of y's helps the x population grow. Hence, x is the predator. Similarly, the term $-\delta xy$ in the dy/dt equation implies that when $x > 0$, y's grow more slowly, so y is the prey. If $y = 0$, then $dx/dt < 0$, so the predators will die out; thus, they must have insufficient alternative food sources. The prey has no limits on its growth other than the predator since, if $x = 0$, then $dy/dt > 0$ and the population increases exponentially.

(b) Since $-\beta xy$ is negative and $+\delta xy$ is positive, x suffers due to its interaction with y and y benefits from its interaction with x. Hence, x is the prey and y is the predator. The predator has other sources of food than the prey since $dy/dt > 0$ even if $x = 0$. Also, the prey has a limit on its growth due to the $-\alpha x^2/N$ term.

22. **(a)** We consider dx/dt in each system. Setting $y = 0$ yields $dx/dt = 5x$ in system (i) and $dx/dt = x$ in system (ii). If the number x of prey is equal for both systems, dx/dt is larger in system (i). Therefore, the prey in system (i) reproduce faster if there are no predators.

(b) We must see what effect the predators (represented by the y-terms) have on dx/dt in each system. Since the magnitude of the coefficient of the xy-term is larger in system (ii) than in system (i), y has a greater effect on dx/dt in system (ii). Hence the predators have a greater effect on the rate of change of the prey in system (ii).

(c) We must see what effect the prey (represented by the x-terms) have on dy/dt in each system. Since x and y are both nonnegative, it follows that

$$-2y + \tfrac{1}{2}xy < -2y + 6xy,$$

and therefore, if the number of predators is equal for both systems, dy/dt is smaller in system (i). Hence more prey are required in system (i) than in system (ii) to achieve a certain growth rate.

23. **(a)** The independent variable is t, and x and y are dependent variables. Since each xy-term is positive, the presence of either species increases the rate of change of the other. Hence, these species cooperate. The parameter α is the growth-rate parameter for x, and γ is the growth-rate parameter for y. The parameter N represents the carrying capacity for x, but y has no carrying capacity. The parameter β measures the benefit to x of the interaction of the two species, and δ measures the benefit to y of the interaction.

(b) The independent variable is t, and x and y are the dependent variables. Since both xy-terms are negative, these species compete. The parameter γ is the growth-rate coefficient for x, and α is the growth-rate parameter for y. Neither population has a carrying capacity. The parameter δ measures the harm to x caused by the interaction of the two species, and β measures the harm to y caused by the interaction.

1. **(a)** Let's check Bob's solution first. Since $dy/dt = 1$ and

$$\frac{y(t)+1}{t+1} = \frac{t+1}{t+1} = 1,$$

Bob's answer is correct.

Now let's check Glen's solution. Since $dy/dt = 2$ and

$$\frac{y(t)+1}{t+1} = \frac{2t+2}{t+1} = 2,$$

Glen's solution is also correct.

Finally let's check Paul's solution. We have $dy/dt = 2t$ on one hand and

$$\frac{y(t)+1}{t+1} = \frac{t^2-1}{t+1} = t-1$$

on the other. Paul is wrong.

(b) At first glance, they should have seen the equilibrium solution $y(t) = -1$ for all t because $dy/dt = 0$ for any constant function and $y = -1$ implies that

$$\frac{y+1}{t+1} = 0$$

independent of t.

Strictly speaking the differential equation is not defined for $t = -1$, and hence the solutions are not defined for $t = -1$.

5. **(a)** This equation is separable. (It is nonlinear and nonautonomous as well.)

(b) We separate variables and integrate to obtain

$$\int \frac{1}{y^2}\,dy = \int t^2\,dt$$

$$-\frac{1}{y} = \frac{t^3}{3} + c$$

$$y(t) = \frac{-1}{(t^3/3)+c},$$

where c is any real number. This function can also be written in the form

$$y(t) = \frac{-3}{t^3+k}$$

where k is any constant. The constant function $y(t) = 0$ for all t is also a solution of this equation. It is the equilibrium solution at $y = 0$.

7. We separate variables and integrate to obtain

$$\int \frac{dy}{2y+1} = \int dt.$$

We get

$$\frac{1}{2}\ln|2y+1| = t + c$$

$$|2y+1| = c_1 e^{2t},$$

where $c_1 = e^{2c}$. As in Exercise 22, we can drop the absolute value signs by replacing $\pm c_1$ with a new constant k_1. Hence, we have

$$2y + 1 = k_1 e^{2t}$$

$$y = \frac{1}{2}\left(k_1 e^{2t} - 1\right),$$

and letting $k = k_1/2$, $y(t) = ke^{2t} - 1/2$. Note that, for $k = 0$, we get the equilibrium solution.

9. We separate variables and integrate to obtain

$$\int e^y\, dy = \int dt$$

$$e^y = t + c,$$

where c is any constant. We obtain $y(t) = \ln(t + c)$.

11. **(a)** This equation is separable.

(b) We separate variables and integrate to obtain

$$\int \frac{1}{y^2}\, dy = \int (2t+3)\, dt$$

$$-\frac{1}{y} = t^2 + 3t + k$$

$$y(t) = \frac{-1}{t^2 + 3t + k},$$

where k is any constant. The constant function $y(t) = 0$ for all t is also a solution of this equation. It is the equilibrium solution at $y = 0$.

13. First note that the differential equation is not defined if $y = 0$.

In order to separate the variables, we write the equation as

$$\frac{dy}{dt} = \frac{t}{y(t^2+1)}$$

to obtain

$$\int y\,dy = \int \frac{t}{t^2+1}\,dt$$

$$\frac{y^2}{2} = \frac{1}{2}\ln(t^2+1) + c,$$

where c is any constant. So we get

$$y^2 = \ln\left(k(t^2+1)\right),$$

where $k = e^{2c}$ (hence any positive constant). We have

$$y(t) = \pm\sqrt{\ln\left(k(t^2+1)\right)},$$

where k is any positive constant and the sign is determined by the initial condition.

15. First note that the differential equation is not defined for $y = -1/2$. We separate variables and integrate to obtain

$$\int (2y+1)\,dy = \int dt$$

$$y^2 + y = t + k,$$

where k is any constant. So

$$y(t) = \frac{-1 \pm \sqrt{4t+4k+1}}{2} = \frac{-1 \pm \sqrt{4t+c}}{2},$$

where c is any constant and the $\pm$ sign is determined by the initial condition.

We can rewrite the answer in the more simple form

$$y(t) = -\frac{1}{2} \pm \sqrt{t + c_1}$$

where $c_1 = k + 1/4$. If k can be any possible constant, then c_1 can be as well.

17. First of all, the equilibrium solutions are $y = 0$ and $y = 1$. Now suppose $y \neq 0$ and $y \neq 1$. We separate variables to obtain

$$\int \frac{1}{y(1-y)}\,dy = \int dt = t + c,$$

where c is any constant. To integrate, we use partial fractions. Write

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}.$$

We must have $A = 1$ and $-A + B = 0$. Hence, $A = B = 1$ and

$$\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}.$$

Consequently,

$$\int \frac{1}{y(1-y)}\,dy = \ln|y| - \ln|1-y| = \ln\left|\frac{y}{1-y}\right|.$$

After integration, we have

$$\ln\left|\frac{y}{1-y}\right| = t + c$$

$$\left|\frac{y}{1-y}\right| = c_1 e^t,$$

where $c_1 = e^c$ is any positive constant. To remove the absolute value signs, we replace the positive constant c_1 with a constant k that can be any real number and get

$$y(t) = \frac{ke^t}{1 + ke^t},$$

where $k = \pm c_1$. If $k = 0$, we get the first equilibrium solution. The formula $y(t) = ke^t/(1 + ke^t)$ yields all the solutions to the differential equation except for the equilibrium solution $y(t) = 1$.

19. The equation can be written in the form

$$\frac{dv}{dt} = (v+1)(t^2-2),$$

and we note that $v(t) = -1$ for all t is an equilibrium solution. Separating variables and integrating, we obtain

$$\int \frac{dv}{v+1} = \int t^2 - 2\,dt$$

$$\ln|v+1| = \frac{t^3}{3} - 2t + c,$$

where c is any constant. Thus,

$$|v+1| = c_1 e^{-2t+t^3/3},$$

where $c_1 = e^c$. We can dispose of the absolute value signs by allowing the constant c_1 to be any real number. In other words,

$$v(t) = -1 + ke^{-2t+t^3/3},$$

where $k = \pm c_1$. Note that, if $k = 0$, we get the equilibrium solution.

21. The function $y(t) = 0$ for all t is an equilibrium solution.

Suppose $y \neq 0$ and separate variables. We get

$$\int y + \frac{1}{y}\,dy = \int e^t\,dt$$

$$\frac{y^2}{2} + \ln|y| = e^t + c,$$

where c is any real constant. We cannot solve this equation for y, so we leave the expression for y in this implicit form. Note that the equilibrium solution $y = 0$ cannot be obtained from this implicit equation.

23. The constant function $w(t) = 0$ is an equilibrium solution. Suppose $w \neq 0$ and separate variables. We get

$$\int \frac{dw}{w} = \int \frac{dt}{t}$$

$$\ln |w| = \ln |t| + c$$

$$= \ln c_1 |t|,$$

where c is any constant and $c_1 = e^c$. Therefore,

$$|w| = c_1 |t|.$$

We can eliminate the absolute value signs by allowing the constant to assume positive or negative values. We have

$$w = kt,$$

where $k = \pm c_1$. Moreover, if $k = 0$ we get the equilibrium solution.

25. Separating variables and integrating, we have

$$\int \frac{1}{x}\, dx = \int -t\, dt$$

$$\ln |x| = -\frac{t^2}{2} + c$$

$$|x| = k_1 e^{-t^2/2},$$

where $k_1 = e^c$. We can eliminate the absolute value signs by allowing the constant k_1 to be either positive or negative. Thus, the general solution is

$$x(t) = ke^{-t^2/2}$$

where $k = \pm k_1$. Using the initial condition to solve for k, we have

$$\frac{1}{\sqrt{\pi}} = x(0) = ke^0 = k.$$

Therefore,

$$x(t) = \frac{e^{-t^2/2}}{\sqrt{\pi}}.$$

27. Separating variables and integrating, we obtain

$$\int \frac{dy}{y^2} = -\int dt$$

$$-\frac{1}{y} = -t + c.$$

So we get

$$y = \frac{1}{t - c}.$$

Now we need to find the constant c so that $y(0) = 1/2$. To do this we solve

$$\frac{1}{2} = \frac{1}{0 - c}$$

and get $c = -2$. The solution of the initial-value problem is

$$y(t) = \frac{1}{t + 2}.$$

29. We do not need to do any computations to solve this initial-value problem. We know that the constant function $y(t) = 0$ for all t is an equilibrium solution, and it satisfies the initial condition.

31. From Exercise 7, we already know that the general solution is

$$y(t) = ke^{2t} - \tfrac{1}{2},$$

so we need only find the constant k for which $y(0) = 3$. We solve

$$3 = ke^0 - \tfrac{1}{2}$$

for k and obtain $k = 7/2$. The solution of the initial-value problem is

$$y(t) = \tfrac{7}{2}e^{2t} - \tfrac{1}{2}.$$

33. We write the equation in the form

$$\frac{dx}{dt} = \frac{t^2}{x(t^3+1)}$$

and separate variables to obtain

$$\int x\,dx = \int \frac{t^2}{t^3+1}\,dt$$

$$\frac{x^2}{2} = \frac{1}{3}\ln|t^3+1| + c,$$

where c is a constant. Hence,

$$x^2 = \frac{2}{3}\ln|t^3+1| + 2c.$$

The initial condition $x(0) = -2$ implies

$$4 = (-2)^2 = \frac{2}{3}\ln|1| + 2c.$$

Thus, $c = 2$. Solving for $x(t)$, we choose the negative square root because $x(0)$ is negative, and we drop the absolute value sign because $t^3 + 1 > 0$ for t near 0. The result is

$$x(t) = -\sqrt{\tfrac{2}{3}\ln(t^3+1) + 4}.$$

35. We separate variables to obtain

$$\int \frac{dy}{1+y^2} = \int t\,dt$$

$$\arctan y = \frac{t^2}{2} + c,$$

where c is a constant. Hence the general solution is

$$y(t) = \tan\left(\frac{t^2}{2} + c\right).$$

Next we find c so that $y(0) = 1$. Solving

$$1 = \tan\left(\frac{0^2}{2} + c\right)$$

yields $c = \pi/4$, and the solution to the initial-value problem is

$$y(t) = \tan\left(\frac{t^2}{2} + \frac{\pi}{4}\right).$$

36. Separating variables and integrating, we obtain

$$\int (2y+3)\,dy = \int dt$$

$$y^2 + 3y = t + c$$

$$y^2 + 3y - (t+c) = 0.$$

We can use the quadratic formula to obtain

$$y = -\tfrac{3}{2} \pm \sqrt{t + c_1},$$

where $c_1 = c + 9/4$. Since $y(0) = 1 > -3/2$ we take the positive square root and solve

$$1 = y(0) = -\tfrac{3}{2} + \sqrt{c_1},$$

so $c_1 = 25/4$. The solution to the initial-value problem is

$$y(t) = -\tfrac{3}{2} + \sqrt{t + \tfrac{25}{4}}.$$

37. Separating variables and integrating, we have

$$\int \frac{1}{y^2}\,dy = \int 2t + 3t^2\,dt$$

$$-\frac{1}{y} = t^2 + t^3 + c$$

$$y = \frac{-1}{t^2 + t^3 + c}.$$

Using $y(1) = -1$ we have

$$-1 = y(1) = \frac{-1}{1+1+c} = \frac{-1}{2+c},$$

so $c = -1$. The solution to the initial-value problem is

$$y(t) = \frac{-1}{t^2 + t^3 - 1}.$$

39. Let $S(t)$ denote the amount of salt (in pounds) in the bucket at time t (in minutes). We derive a differential equation for S by considering the difference between the rate that salt is entering the bucket and the rate that salt is leaving the bucket. Salt is entering the bucket at the rate of 1/4 pounds per minute. The rate that salt is leaving the bucket is the product of the concentration of salt in the mixture and the rate that the mixture is leaving the bucket. The concentration is $S/5$, and the mixture is leaving the bucket at the rate of $1/2$ gallons per minute. We obtain the differential equation

$$\frac{dS}{dt} = \frac{1}{4} - \frac{S}{5} \cdot \frac{1}{2},$$

which can be rewritten as

$$\frac{dS}{dt} = \frac{5 - 2S}{20}.$$

This differential equation is separable, and we can find the general solution by integrating

$$\int \frac{1}{5 - 2S}\, dS = \int \frac{1}{20}\, dt.$$

We have

$$-\frac{\ln|5 - 2S|}{2} = \frac{t}{20} + c$$

$$\ln|5 - 2S| = -\frac{t}{10} + c_1$$

$$|5 - 2S| = c_2 e^{-t/10}.$$

We can eliminate the absolute value signs and determine c_2 using the initial condition $S(0) = 0$ (the water is initially free of salt). We have $c_2 = 5$, and the solution is

$$S(t) = 2.5 - 2.5e^{-t/10} = 2.5(1 - e^{-t/10}).$$

(a) When $t = 1$, we have $S(1) = 2.5(1 - e^{-0.1}) \approx 0.238$ lbs.

(b) When $t = 10$, we have $S(10) = 2.5(1 - e^{-1}) \approx 1.58$ lbs.

(c) When $t = 60$, we have $S(60) = 2.5(1 - e^{-6}) \approx 2.49$ lbs.

(d) When $t = 1000$, we have $S(1000) = 2.5(1 - e^{-100}) \approx 2.50$ lbs.

(e) When t is very large, the $e^{-t/10}$ term is close to zero, so $S(t)$ is very close to 2.5 lbs. In this case, we can also reach the same conclusion by doing a qualitative analysis of the solutions of the equation. The constant solution $S(t) = 2.5$ is the only equilibrium solution for this equation, and by examining the sign of dS/dt, we see that all solutions approach $S = 2.5$ as t increases.

40. Rewrite the equation as

$$\frac{dC}{dt} = -k_1 C + (k_1 N + k_2 E),$$

separate variables, and integrate to obtain

$$\int \frac{1}{-k_1 C + (k_1 N + k_2 E)}\, dC = \int dt$$

$$-\frac{1}{k_1} \ln|-k_1 C + k_1 N + k_2 E| = t + c$$

$$-k_1 C + k_1 N + k_2 E = c_1 e^{-k_1 t},$$

where c_1 is a constant determined by the initial condition. Hence,

$$C(t) = N + \frac{k_2}{k_1} E - c_2 e^{-k_1 t},$$

where c_2 is a constant.

(a) Substituting the given values for the parameters, we obtain

$$C(t) = 600 - c_2 e^{-0.1t},$$

and the initial condition $C(0) = 150$ gives $c_2 = 450$, which implies that

$$C(t) = 600 - 450 e^{-0.1t}.$$

Hence, $C(2) \approx 232$.

(b) Using part (a), $C(5) \approx 328$.

(c) When t is very large, $e^{-0.1t}$ is very close to zero, so $C(t) \approx 600$. (We could also obtain this conclusion by doing a qualitative analysis of the solutions.)

(d) Using the new parameter values and $C(0) = 600$ yields

$$C(t) = 300 + 300 e^{-0.1t},$$

so $C(1) \approx 571$, $C(5) \approx 482$, and $C(t) \to 300$ as $t \to \infty$.

(e) Again changing the parameter values and using $C(0) = 600$, we have

$$C(t) = 500 + 100 e^{-0.1t},$$

so $C(1) \approx 590$, $C(5) \approx 560$, and $C(t) \to 500$ as $t \to \infty$.

41. **(a)** If we let k denote the proportionality constant in Newton's law of cooling, the differential equation satisfied by the temperature T of the chocolate is

$$\frac{dT}{dt} = k(T - 70).$$

We also know that $T(0) = 170$ and that $dT/dt = -20$ at $t = 0$. Therefore, we obtain k by evaluating the differential equation at $t = 0$. We have

$$-20 = k(170 - 70),$$

so $k = -0.2$. The initial-value problem is

$$\frac{dT}{dt} = -0.2(T - 70), \quad T(0) = 170.$$

(b) We can solve the initial-value problem in part (a) by separating variables. We have

$$\int \frac{dT}{T - 70} = \int -0.2\,dt$$

$$\ln|T - 70| = -0.2t + k$$

$$|T - 70| = ce^{-0.2t}.$$

Since the temperature of the chocolate cannot become lower than the temperature of the room, we can ignore the absolute value and conclude

$$T(t) = 70 + ce^{-0.2t}.$$

Now we use the initial condition $T(0) = 170$ to find the constant c because

$$170 = T(0) = 70 + ce^{-0.2(0)},$$

which implies that $c = 100$. The solution is

$$T = 70 + 100e^{-0.2t}.$$

In order to find t so that the temperature is 110° F, we solve

$$110 = 70 + 100e^{-0.2t}$$

for t obtaining

$$\tfrac{2}{5} = e^{-0.2t}$$

$$\ln \tfrac{2}{5} = -0.2t$$

so that

$$t = \frac{\ln(2/5)}{-0.2} \approx 4.6.$$

43. **(a)** We rewrite the differential equation as

$$\frac{dv}{dt} = g\left(1 - \frac{k}{mg}v^2\right).$$

Letting $\alpha = \sqrt{k/(mg)}$ and separating variables, we have

$$\int \frac{dv}{1-\alpha^2 v^2} = \int g\,dt.$$

Now we use the partial fractions decomposition

$$\frac{1}{1-\alpha^2 v^2} = \frac{1/2}{1+\alpha v} + \frac{1/2}{1-\alpha v}$$

to obtain

$$\int \frac{dv}{1+\alpha v} + \int \frac{dv}{1-\alpha v} = 2gt + c,$$

where c is an arbitrary constant. Integrating the left-hand side, we get

$$\frac{1}{\alpha}\Big(\ln|1+\alpha v| - \ln|1-\alpha v|\Big) = 2gt + c.$$

Multiplying through by α and using the properties of logarithms, we have

$$\ln\left|\frac{1+\alpha v}{1-\alpha v}\right| = 2\alpha g t + c.$$

Exponentiating and eliminating the absolute value signs yields

$$\frac{1+\alpha v}{1-\alpha v} = Ce^{2\alpha g t}.$$

Solving for v, we have

$$v = \frac{1}{\alpha}\frac{Ce^{2\alpha g t} - 1}{Ce^{2\alpha g t} + 1}.$$

Recalling that $\alpha = \sqrt{k/(mg)}$, we see that $\alpha g = \sqrt{kg/m}$, and we get

$$v(t) = \sqrt{\frac{mg}{k}}\left(\frac{Ce^{2\sqrt{(kg/m)}\,t} - 1}{Ce^{2\sqrt{(kg/m)}\,t} + 1}\right).$$

Note: If we assume that $v(0) = 0$, then $C = 1$. The solution to this initial-value problem is often expressed in terms of the hyperbolic tangent function as

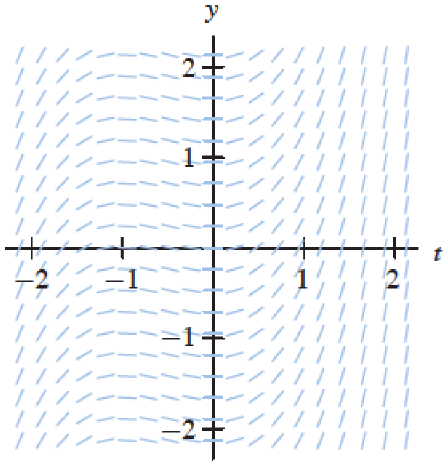
$$v = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}\,t\right).$$

(b) The fraction in the parentheses of the general solution

$$v(t) = \sqrt{\frac{mg}{k}}\left(\frac{Ce^{2\sqrt{(kg/m)}\,t} - 1}{Ce^{2\sqrt{(kg/m)}\,t} + 1}\right),$$

tends to 1 as $t \to \infty$, so the limit of $v(t)$ as $t \to \infty$ is $\sqrt{mg/k}$.

1.



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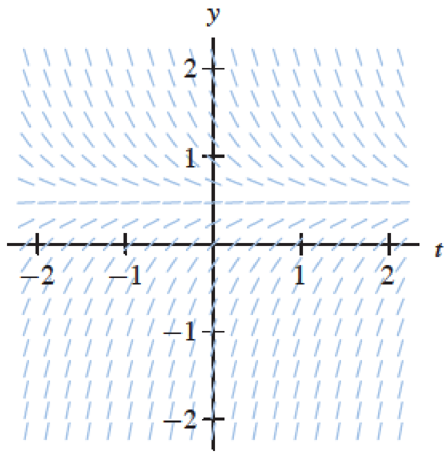

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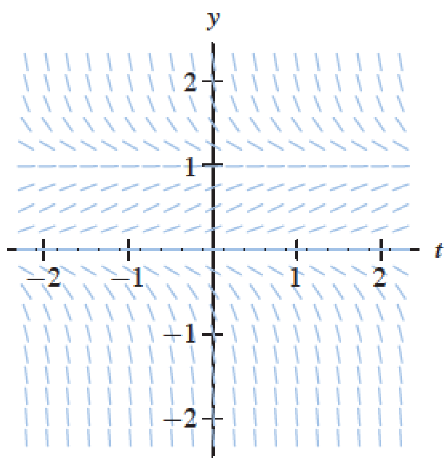
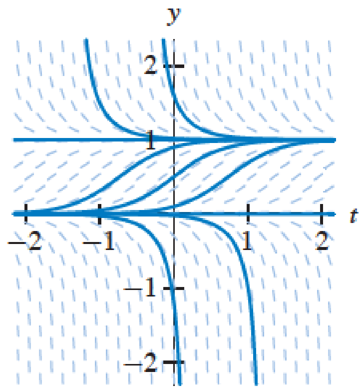
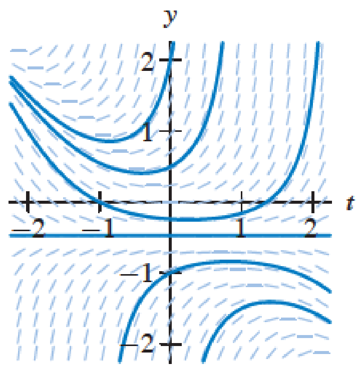


7. **(a)**



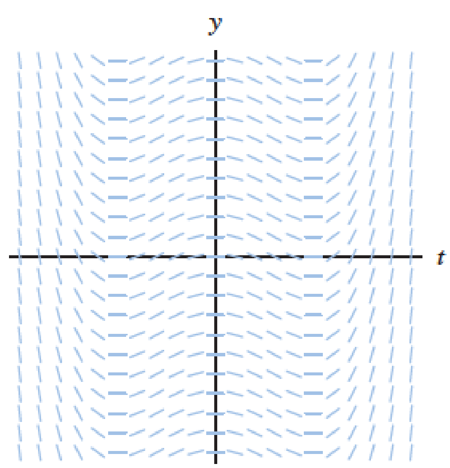
(b) The solution with $y(0) = 1/2$ approaches the equilibrium value $y = 1$ from below as t increases. It decreases toward $y = 0$ as t decreases.

9. **(a)**

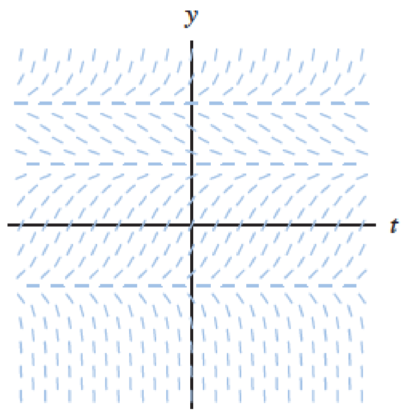


(b) The solution $y(t)$ with $y(0) = 1/2$ has $y(t) \to \infty$ both as t increases and as t decreases.

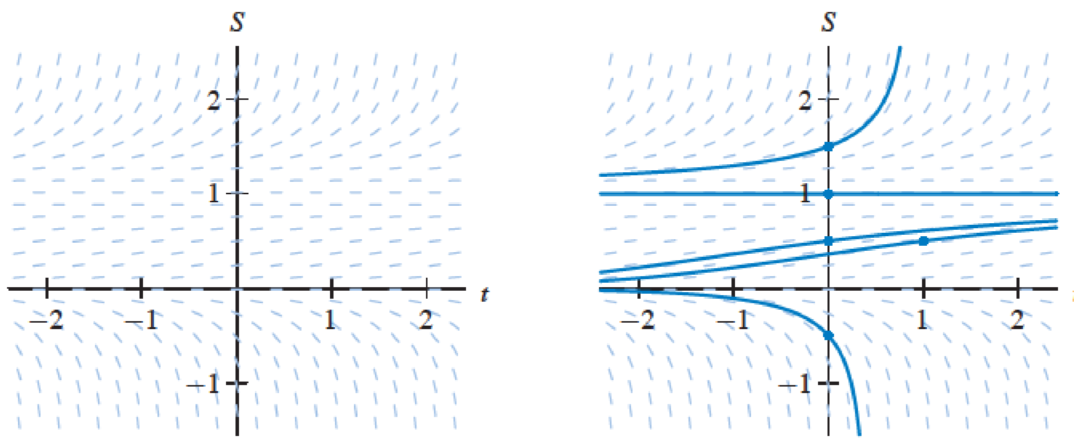
13. The slope field in the ty-plane is constant along vertical lines.



14. Because f depends only on y (the equation is autonomous), the slope field is constant along horizontal lines in the ty-plane. The roots of f correspond to equilibrium solutions. If $f(y) > 0$, the corresponding lines in the slope field have positive slope. If $f(y) < 0$, the corresponding lines in the slope field have negative slope.



15.



16. **(a)** This slope field is constant along horizontal lines, so it corresponds to an autonomous equation. The autonomous equations are (i), (ii), and (iii). This field does not correspond to equation (ii) because it has the equilibrium solution $y = -1$. The slopes are negative for $y < -1$. Consequently, this field corresponds to equation (iii).

(b) Note that the slopes are constant along vertical lines—lines along which t is constant, so the right-hand side of the corresponding equation depends only on t. The only choices are equations (iv) and (viii). Since the slopes are negative for $-\sqrt{2} < t < \sqrt{2}$, this slope field corresponds to equation (viii).

(c) This slope field depends both on y and on t, so it can only correspond to equations (v), (vi), or (vii). Since this field has the equilibrium solution $y = 0$, this slope field corresponds to equation (v).

(d) This slope field also depends on both y and on t, so it can only correspond to equations (v), (vi), or (vii). This field does not correspond to equation (v) because $y = 0$ is not an equilibrium solution. Since the slopes are nonnegative for $y > -1$, this slope field corresponds to equation (vi).