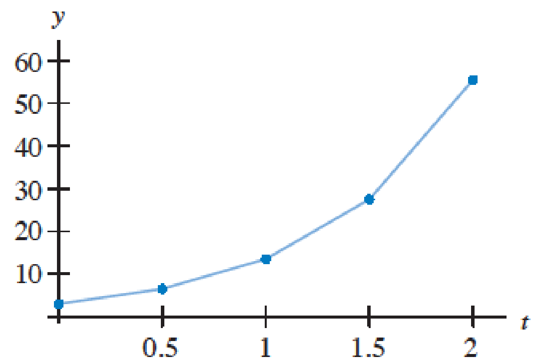
**1.**

Table 1.1
Results of Euler's method

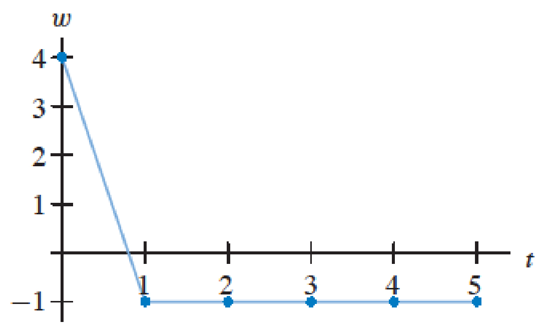
| $k$ | $t_k$ | $y_k$ | $m_k$ |
|---|---|---|---|
| 0 | 0 | 3 | 7 |
| 1 | 0.5 | 6.5 | 14 |
| 2 | 1.0 | 13.5 | 28 |
| 3 | 1.5 | 27.5 | 56 |
| 4 | 2.0 | 55.5 | |




**5.**

Table 1.5
Results of Euler's method

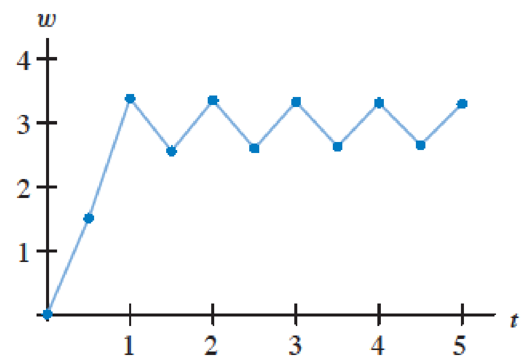
| $k$ | $t_k$ | $w_k$ | $m_k$ |
|---|---|---|---|
| 0 | 0 | 4 | −5 |
| 1 | 1 | −1 | 0 |
| 2 | 2 | −1 | 0 |
| 3 | 3 | −1 | 0 |
| 4 | 4 | −1 | 0 |
| 5 | 5 | −1 | |




**6.**

Table 1.6
Results of Euler's method (shown rounded to two decimal places)

| $k$ | $t_k$ | $w_k$ | $m_k$ |
|---|---|---|---|
| 0 | 0 | 0 | 3 |
| 1 | 0.5 | 1.5 | 3.75 |
| 2 | 1.0 | 3.38 | −1.64 |
| 3 | 1.5 | 2.55 | 1.58 |
| 4 | 2.0 | 3.35 | −1.50 |
| 5 | 2.5 | 2.59 | 1.46 |
| 6 | 3.0 | 3.32 | −1.40 |
| 7 | 3.5 | 2.62 | 1.36 |
| 8 | 4.0 | 3.31 | −1.31 |
| 9 | 4.5 | 2.65 | 1.28 |
| 10 | 5.0 | 3.29 | |




**11.** As the solution approaches the equilibrium solution corresponding to $w = 3$, its slope decreases. We do not expect the solution to "jump over" an equilibrium solution (see the Existence and Uniqueness Theorem in Section 1.5).

**14.** Euler's method is not accurate in either case because the step size is too large. In Exercise 5, the approximate solution "jumps onto" an equilibrium solution. In Exercise 6, the approximate solution "crisscrosses" a different equilibrium solution. Approximate solutions generated with smaller values of $\Delta t$ indicate that the actual solutions do not exhibit this behavior (see the Existence and Uniqueness Theorem of Section 1.5).

**1.** Since the constant function $y_1(t) = 3$ for all $t$ is a solution, then the graph of any other solution $y(t)$ with $y(0) < 3$ cannot cross the line $y = 3$ by the Uniqueness Theorem. So $y(t) < 3$ for all $t$ in the domain of $y(t)$.

**3.** Because $y_2(0) < y(0) < y_1(0)$, we know that

$$-t^2 = y_2(t) < y(t) < y_1(t) = t + 2$$

for all $t$. This restricts how large positive or negative $y(t)$ can be for a given value of $t$ (that is, between $-t^2$ and $t + 2$). As $t \to -\infty$, $y(t) \to -\infty$ between $-t^2$ and $t + 2$ ($y(t) \to -\infty$ as $t \to -\infty$ at least linearly, but no faster than quadratically).

**5.** The Existence Theorem implies that a solution with this initial condition exists, at least for a small $t$-interval about $t = 0$. This differential equation has equilibrium solutions $y_1(t) = 0$, $y_2(t) = 1$, and $y_3(t) = 3$ for all $t$. Since $y(0) = 4$, the Uniqueness Theorem implies that $y(t) > 3$ for all $t$ in the domain of $y(t)$. Also, $dy/dt > 0$ for all $y > 3$, so the solution $y(t)$ is increasing for all $t$ in its domain. Finally, $y(t) \to 3$ as $t \to -\infty$.

**7.** The Existence Theorem implies that a solution with this initial condition exists, at least for a small $t$-interval about $t = 0$. Because $1 < y(0) < 3$ and $y_1(t) = 1$ and $y_2(t) = 3$ are equilibrium solutions of the differential equation, we know that the solution exists for all $t$ and that $1 < y(t) < 3$ for all $t$ by the Uniqueness Theorem. Also, $dy/dt < 0$ for $1 < y < 3$, so $dy/dt$ is always negative for this solution. Hence, $y(t) \to 1$ as $t \to \infty$, and $y(t) \to 3$ as $t \to -\infty$.

**9.** **(a)** To check that $y_1(t) = t^2$ is a solution, we compute

$$\frac{dy_1}{dt} = 2t$$

and

$$\begin{aligned} -y_1^2 + y_1 + 2y_1t^2 + 2t - t^2 - t^4 &= -(t^2)^2 + (t^2) + 2(t^2)t^2 + 2t - t^2 - t^4 \\ &= 2t. \end{aligned}$$

To check that $y_2(t) = t^2 + 1$ is a solution, we compute

$$\frac{dy_2}{dt} = 2t$$

and

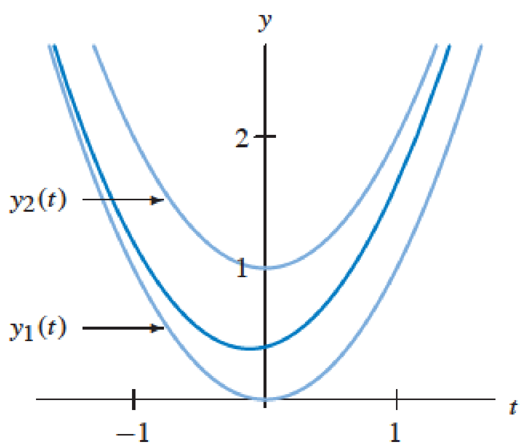
$$\begin{aligned} -y_2^2 + y_2 + 2y_2t^2 + 2t - t^2 - t^4 &= -(t^2+1)^2 + (t^2+1) + 2(t^2+1)t^2 \\ &\quad + 2t - t^2 - t^4 \\ &= 2t. \end{aligned}$$

**(b)** The initial values of the two solutions are $y_1(0) = 0$ and $y_2(0) = 1$. Thus if $y(t)$ is a solution and $y_1(0) = 0 < y(0) < 1 = y_2(0)$, then we can apply the Uniqueness Theorem to obtain

$$y_1(t) = t^2 < y(t) < t^2 + 1 = y_2(t)$$

for all $t$. Note that since the differential equation satisfies the hypothesis of the Existence and Uniqueness Theorem over the entire $ty$-plane, we can continue to extend the solution as long as it does not escape to $\pm\infty$ in finite time. Since it is bounded above and below by solutions that exist for all time, $y(t)$ is defined for all time also.

**(c)**



**11.** The key observation is that the differential equation is not defined when $t = 0$.

**(a)** Note that $dy_1/dt = 0$ and $y_1/t^2 = 0$, so $y_1(t)$ is a solution.

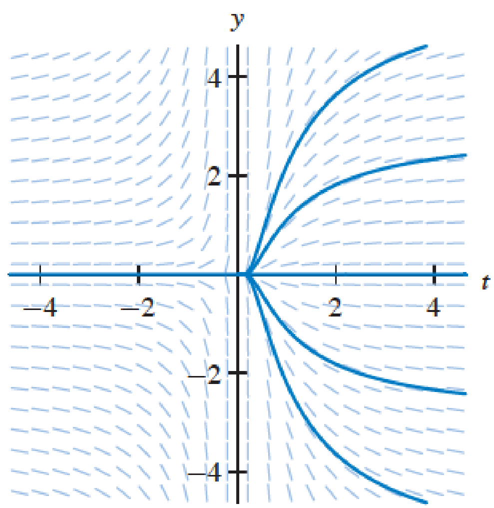
**(b)** Separating variables, we have

$$\int \frac{dy}{y} = \int \frac{dt}{t^2}.$$

Solving for $y$ we obtain $y(t) = ce^{-1/t}$, where $c$ is any constant. Thus, for any real number $c$, define the function $y_c(t)$ by

$$y_c(t) = \begin{cases} 0 & \text{for } t \le 0; \\ ce^{-1/t} & \text{for } t > 0. \end{cases}$$

For each $c$, $y_c(t)$ satisfies the differential equation for all $t \neq 0$.



There are infinitely many solutions of the form $y_c(t)$ that agree with $y_1(t)$ for $t < 0$.

**(c)** Note that $f(t, y) = y/t^2$ is not defined at $t = 0$. Therefore, we *cannot* apply the Uniqueness Theorem for the initial condition $y(0) = 0$. The "solution" $y_c(t)$ given in part (b) actually represents two solutions, one for $t < 0$ and one for $t > 0$.

**13.** **(a)** The equation is separable. We separate the variables and compute

$$\int y^{-3}\,dy = \int dt.$$

Solving for $y$, we obtain

$$y(t) = \frac{1}{\sqrt{c - 2t}}$$

for any constant $c$. To find the desired solution, we use the initial condition $y(0) = 1$ and obtain $c = 1$. So the solution to the initial-value problem is

$$y(t) = \frac{1}{\sqrt{1 - 2t}}.$$

**(b)** This solution is defined when $-2t + 1 > 0$, which is equivalent to $t < 1/2$.

**(c)** As $t \to 1/2^-$, the denominator of $y(t)$ becomes a small positive number, so $y(t) \to \infty$. We only consider $t \to 1/2^-$ because the solution is defined only for $t < 1/2$. (The other "branch" of the function is also a solution, but the solution that includes $t = 0$ in its domain is not defined for $t \geq 1/2$.) As $t \to -\infty$, $y(t) \to 0$.

**17.** This exercise shows that solutions of autonomous equations cannot have local maximums or minimums. Hence they must be either constant or monotonically increasing or monotonically decreasing. A useful corollary is that a function $y(t)$ that oscillates cannot be the solution of an autonomous differential equation.

**(a)** Note $dy_1/dt = 0$ at $t = t_0$ because $y_1(t)$ has a local maximum. Because $y_1(t)$ is a solution, we know that $dy_1/dt = f(y_1(t))$ for all $t$ in the domain of $y_1(t)$. In particular,

$$0 = \left.\frac{dy_1}{dt}\right|_{t=t_0} = f(y_1(t_0)) = f(y_0),$$

so $f(y_0) = 0$.

**(b)** This differential equation is autonomous, so the slope marks along any given horizontal line are parallel. Hence, the slope marks along the line $y = y_0$ must all have zero slope.

**(c)** For all $t$,

$$\frac{dy_2}{dt} = \frac{d(y_0)}{dt} = 0$$

because the derivative of a constant function is zero, and for all $t$

$$f(y_2(t)) = f(y_0) = 0.$$

So $y_2(t)$ is a solution.

**(d)** By the Uniqueness Theorem, we know that two solutions that are in the same place at the same time are the same solution. We have $y_1(t_0) = y_0 = y_2(t_0)$. Moreover, $y_1(t)$ is assumed to be a solution, and we showed that $y_2(t)$ is a solution in parts (a) and (b) of this exercise. So $y_1(t) = y_2(t)$ for all $t$. In other words, $y_1(t) = y_0$ for all $t$.

**(e)** Follow the same four steps as before. We still have $dy_1/dt = 0$ at $t = t_0$ because $y_1$ has a local minimum at $t = t_0$.

**18.** **(a)** Solving for $r$, we get

$$r = \left(\frac{3v}{4\pi}\right)^{1/3}.$$

Consequently,

$$\begin{aligned} s(t) &= 4\pi \left(\frac{3v}{4\pi}\right)^{2/3} \\ &= cv(t)^{2/3}, \end{aligned}$$

where $c$ is a constant. Since we are assuming that the rate of growth of $v(t)$ is proportional to its surface area $s(t)$, we have

$$\frac{dv}{dt} = kv^{2/3},$$

where $k$ is a constant.

**(b)** The partial derivative with respect to $v$ of $dv/dt$ does not exist at $v = 0$. Hence the Uniqueness Theorem tells us nothing about the uniqueness of solutions that involve $v = 0$. In fact, if we use the techniques described in the section related to the uniqueness of solutions for $dy/dt = 3y^{2/3}$, we can find infinitely many solutions with this initial condition.

**(c)** Since it does not make sense to talk about rain drops with negative volume, we always have $v \geq 0$. Once $v > 0$, the evolution of the drop is completely determined by the differential equation.

What is the physical significance of a drop with $v = 0$? It is tempting to interpret the fact that solutions can have $v = 0$ for an arbitrary amount of time before beginning to grow as a statement that the rain drops can spontaneously begin to grow at any time. Since the model gives no information about when a solution with $v = 0$ starts to grow, it is not very useful for the understanding the initial formation of rain drops. The safest assertion is to say is the model breaks down if $v = 0$.