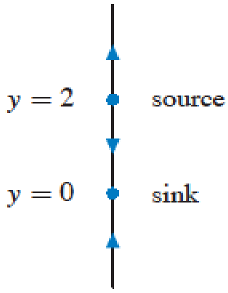
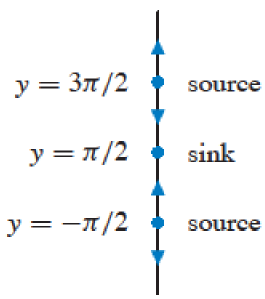
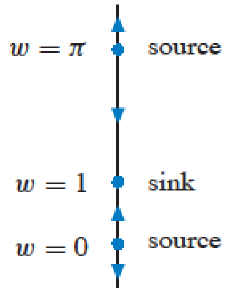
1. The equilibrium points of $dy/dt = f(y)$ are the numbers y where $f(y) = 0$. For $f(y) = 3y(y-2)$, the equilibrium points are $y = 0$ and $y = 2$. Since $f(y)$ is positive for $y < 0$, negative for $0 < y < 2$, and positive for $y > 2$, the equilibrium point $y = 0$ is a sink and the equilibrium point $y = 2$ is a source.



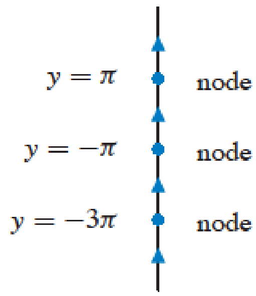
3. The equilibrium points of $dy/dt = f(y)$ are the numbers y where $f(y) = 0$. For $f(y) = \cos y$, the equilibrium points are $y = \pi/2 + n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$. Since $\cos y > 0$ for $-\pi/2 < y < \pi/2$ and $\cos y < 0$ for $\pi/2 < y < 3\pi/2$, we see that the equilibrium point at $y = \pi/2$ is a sink. Since the sign of $\cos y$ alternates between positive and negative in a period fashion, we see that the equilibrium points at $y = \pi/2 + 2n\pi$ are sinks and the equilibrium points at $y = 3\pi/2 + 2n\pi$ are sources.



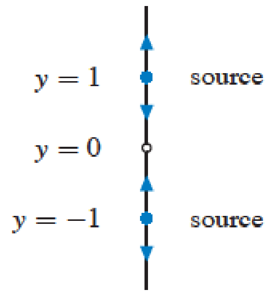
5. The equilibrium points of $dw/dt = f(w)$ are the numbers w where $f(w) = 0$. For $f(w) = (1 - w)\sin w$, the equilibrium points are $w = 1$ and $w = n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$. The sign of $(1 - w)\sin w$ alternates between positive and negative at successive zeros. It is negative for $-\pi < w < 0$ and positive for $0 < w < 1$. Therefore, $w = 0$ is a source, and the equilibrium points alternate between sinks and sources.



9. The equilibrium points of $dy/dt = f(y)$ are the numbers y where $f(y) = 0$. For $f(y) = 1 + \cos y$, the equilibrium points are $y = n\pi$, where $n = \pm 1, \pm 3, \ldots$. Since $f(y)$ is non-negative for all values of y, all of the equilibrium points are nodes.



11. The equilibrium points of $dy/dt = f(y)$ are the numbers y where $f(y) = 0$. For $f(y) = y \ln |y|$, there are equilibrium points at $y = \pm 1$. In addition, although the function $f(y)$ is technically undefined at $y = 0$, the limit of $f(y)$ as $y \to 0$ is 0. Thus we can treat $y = 0$ as another equilibrium point. Since $f(y) < 0$ for $y < -1$ and $0 < y < 1$, and $f(y) > 0$ for $y > 1$ and $-1 < y < 0$, $y = -1$ is a source, $y = 0$ is a sink, and $y = 1$ is a source.



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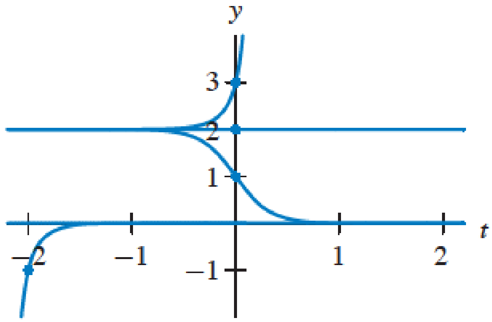

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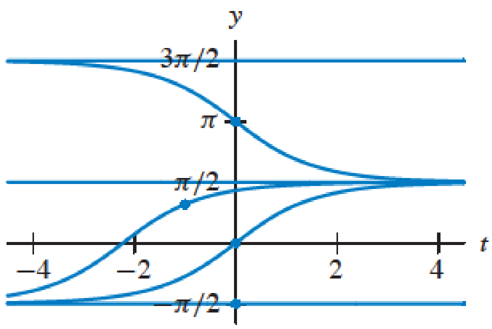

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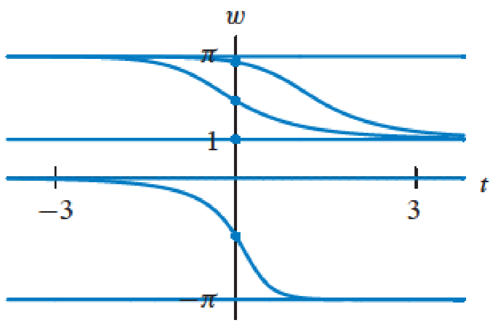

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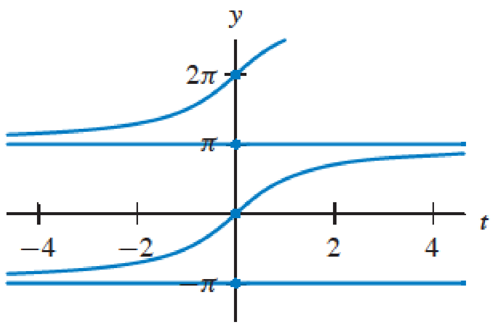


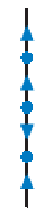
23. The initial value $y(0) = 2$ is between the equilibrium points $y = 2 - \sqrt{2}$ and $y = 2 + \sqrt{2}$. Also, $dy/dt < 0$ for $2-\sqrt{2} < y < 2+\sqrt{2}$. Hence the solution is decreasing and tends toward $y = 2-\sqrt{2}$ as $t \to \infty$. It tends toward $y = 2 + \sqrt{2}$ as $t \to -\infty$.

25. The initial value $y(0) = -4$ is below both equilibrium points. Since $dy/dt > 0$ for $y < 2 - \sqrt{2}$, the solution is increasing for all t and tends to the equilibrium point $y = 2 - \sqrt{2}$ as $t \to \infty$. As t decreases, $y(t) \to -\infty$ in finite time.

27. The initial value $y(3) = 1$ is between the equilibrium points $y = 2 - \sqrt{2}$ and $y = 2 + \sqrt{2}$. Also, $dy/dt < 0$ for $2-\sqrt{2} < y < 2+\sqrt{2}$. Hence the solution is decreasing and tends toward the smaller equilibrium point $y = 2 - \sqrt{2}$ as $t \to \infty$. It tends toward the larger equilibrium point $y = 2 + \sqrt{2}$ as $t \to -\infty$.

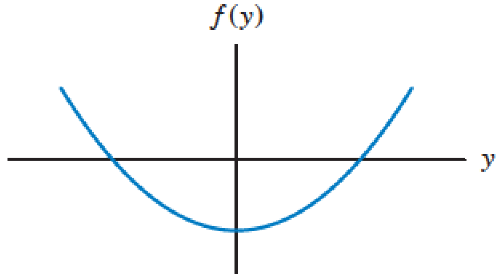
29. The function $f(y)$ has two zeros $\pm y_0$, where y_0 is some positive number. So the differential equation $dy/dt = f(y)$ has two equilibrium solutions, one for each zero. Also, $f(y) < 0$ if $-y_0 < y < y_0$ and $f(y) > 0$ if $y < -y_0$ or if $y > y_0$. Hence y_0 is a source and $-y_0$ is a sink.

31. The function $f(y)$ has three zeros. We denote them as y_1, y_2, and y_3, where $y_1 < 0 < y_2 < y_3$. So the differential equation $dy/dt = f(y)$ has three equilibrium solutions, one for each zero. Also, $f(y) > 0$ if $y < y_1$, $f(y) < 0$ if $y_1 < y < y_2$, and $f(y) > 0$ if $y_2 < y < y_3$ or if $y > y_3$. Hence y_1 is a sink, y_2 is a source, and y_3 is a node.



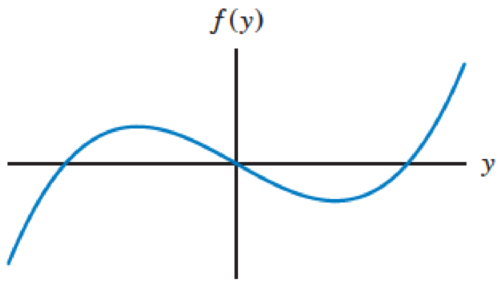
33. Since there are two equilibrium points, the graph of $f(y)$ must touch the y-axis at two distinct numbers y_1 and y_2. Assume that $y_1 < y_2$. Since the arrows point up if $y < y_1$ and if $y > y_2$, we must have $f(y) > 0$ for $y < y_1$ and for $y > y_2$. Similarly, $f(y) < 0$ for $y_1 < y < y_2$.

The precise location of the equilibrium points is not given, and the direction of the arrows on the phase line is determined only by the sign (and not the magnitude) of $f(y)$. So the following graph is one of many possible answers.



35. Since there are three equilibrium points (one appearing to be at $y = 0$), the graph of $f(y)$ must touch the y-axis at three numbers y_1, y_2, and y_3. We assume that $y_1 < y_2 = 0 < y_3$. Since the arrows point down for $y < y_1$ and $y_2 < y < y_3$, $f(y) < 0$ for $y < y_1$ and for $y_2 < y < y_3$. Similarly, $f(y) > 0$ if $y_1 < y < y_2$ and if $y > y_3$.

The precise location of the equilibrium points is not given, and the direction of the arrows on the phase line is determined only by the sign (and not the magnitude) of $f(y)$. So the following graph is one of many possible answers.



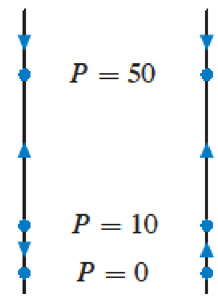
37. **(a)** This phase line has two equilibrium points, $y = 0$ and $y = 1$. Equations (ii), (iv), (vi), and (viii) have exactly these equilibria. There exists a node at $y = 0$. Only equations (iv) and (viii) have a node at $y = 0$. Moreover, for this phase line, $dy/dt < 0$ for $y > 1$. Only equation (viii) satisfies this property. Consequently, the phase line corresponds to equation (viii).

(b) This phase line has two equilibrium points, $y = 0$ and $y = 1$. Equations (ii), (iv), (vi) and (viii) have exactly these equilibria. Moreover, for this phase line, $dy/dt > 0$ for $y > 1$. Only equations (iv) and (vi) satisfy this property. Lastly, $dy/dt > 0$ for $y < 0$. Only equation (vi) satisfies this property. Consequently, the phase line corresponds to equation (vi).

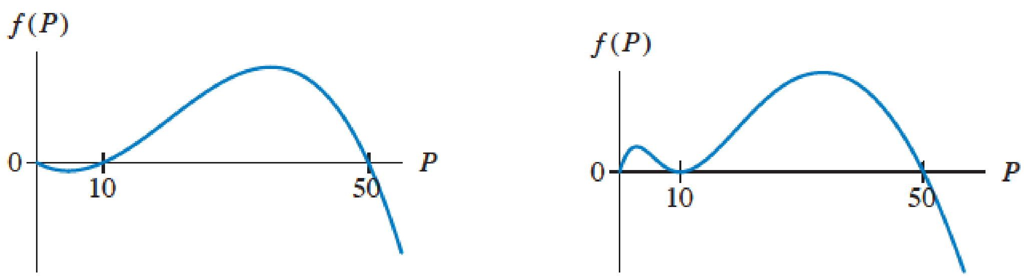
(c) This phase line has an equilibrium point at $y = 3$. Only equations (i) and (v) have this equilibrium point. Moreover, this phase line has another equilibrium point at $y = 0$. Only equation (i) satisfies this property. Consequently, the phase line corresponds to equation (i).

(d) This phase line has an equilibrium point at $y = 2$. Only equations (iii) and (vii) have this equilibrium point. Moreover, there exists a node at $y = 0$. Only equation (vii) satisfies this property. Consequently, the phase line corresponds to equation (vii).

39. **(a)** In terms of the phase line with $P \geq 0$, there are three equilibrium points. If we assume that $f(P)$ is differentiable, then a decreasing population at $P = 100$ implies that $f(P) < 0$ for $P > 50$. An increasing population at $P = 25$ implies that $f(P) > 0$ for $10 < P < 50$. These assumptions leave two possible phase lines since the arrow between $P = 0$ and $P = 10$ is undetermined.



(b) Given the observations in part (a), we see that there are two basic types of graphs that go with the assumptions. However, there are many graphs that correspond to each possibility. The following two graphs are representative.

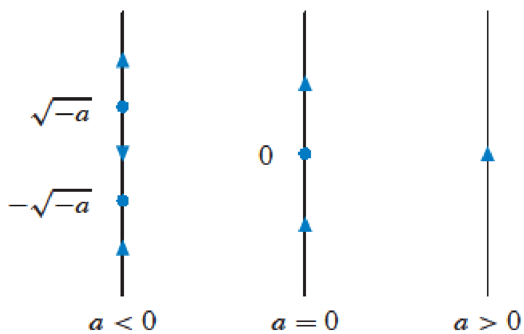


(c) The functions $f(P) = P(P - 10)(50 - P)$ and $f(P) = P(P - 10)^2(50 - P)$ respectively are two examples but there are many others.

41. The equilibrium points occur at solutions of $dy/dt = y^2 + a = 0$. For $a > 0$, there are no equilibrium points. For $a = 0$, there is one equilibrium point, $y = 0$. For $a < 0$, there are two equilibrium points, $y = \pm\sqrt{-a}$.

To draw the phase lines, note that:

- If $a > 0$, $dy/dt = y^2 + a > 0$, so the solutions are always increasing.
- If $a = 0$, $dy/dt > 0$ unless $y = 0$. Thus, $y = 0$ is a node.
- For $a < 0$, $dy/dt < 0$ for $-\sqrt{-a} < y < \sqrt{-a}$, and $dy/dt > 0$ for $y < -\sqrt{-a}$ and for $y > \sqrt{-a}$.



(a) The phase lines for $a < 0$ are qualitatively the same, and the phase lines for $a > 0$ are qualitatively the same.

(b) The phase line undergoes a qualitative change at $a = 0$.

44. **(a)** The differential equation is not defined for $y = -1$ and $y = 2$ and has no equilibria. So the phase line has holes at $y = -1$ and $y = 2$. The function $f(y) = 1/((y-2)(y+1))$ is positive for $y > 2$ and for $y < -1$. It is negative for $-1 < y < 2$. Thus, the phase line to the right corresponds to this differential equation.

Since the value, 1/2, of the initial condition $y(0) = 1/2$ is in the interval where the function $f(y)$ is negative, the solution is decreasing. It reaches $y = -1$ in finite time. As t decreases, the solution reaches $y = 2$ in finite time. Strictly speaking, the solution does not continue beyond the values $y = -1$ and $y = 2$ because the differential equation is not defined for $y = -1$ and $y = 2$.

(b) We can solve the differential equation analytically. We separate variables and integrate. We get

$$\int (y-2)(y+1)\, dy = \int dt$$

$$\frac{y^3}{3} - \frac{y^2}{2} - 2y = t + c,$$

where c is a constant. Using $y(0) = 1/2$, we get $c = 13/12$. Therefore the solution to the initial-value problem is the unique solution $y(t)$ that satisfies the equation

$$4y^3 - 6y^2 - 24y - 24t + 13 = 0$$

with $-1 < y(t) < 2$. It is not easy to solve this equation explicitly. However, in order to obtain the domain of this solution, we substitute $y = -1$ and $y = 2$ into the equation, and we get $t = -9/8$ and $t = 9/8$ respectively.

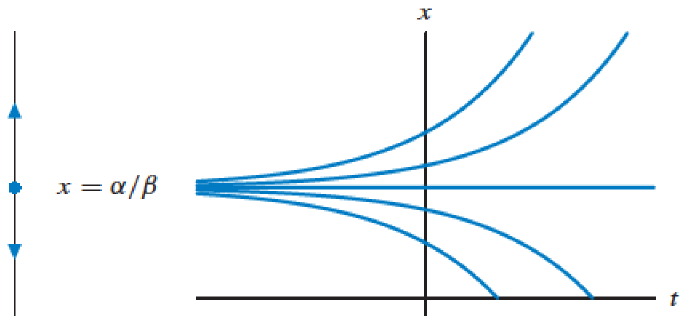
45. One assumption of the model is that, if no people are present, then the time between trains decreases at a constant rate. Hence the term $-\alpha$ represents this assumption. The parameter α should be positive, so that $-\alpha$ makes a negative contribution to dx/dt.

The term βx represents the effect of the passengers. The parameter β should be positive so that βx contributes positively to dx/dt.

46. **(a)** Solving $\beta x - \alpha = 0$, we see that the equilibrium point is $x = \alpha/\beta$.

(b) Since $f(x) = \beta x - \alpha$ is positive for $x > \alpha/\beta$ and negative for $x < \alpha/\beta$, the equilibrium point is a source.

(c) and **(d)**



(e) We separate the variables and integrate to obtain

$$\int \frac{dx}{\beta x - \alpha} = \int dt$$

$$\frac{1}{\beta} \ln |\beta x - \alpha| = t + c,$$

which yields the general solution $x(t) = \alpha/\beta + ke^{\beta t}$, where k is any constant.

47. Note that the only equilibrium point is a source. If the initial gap between trains is too large, then x will increase without bound. If it is too small, x will decrease to zero. When $x = 0$, the two trains are next to each other, and they will stay together since $x < 0$ is not physically possible in this problem.

If the time between trains is exactly the equilibrium value ($x = \alpha/\beta$), then theoretically $x(t)$ is constant. However, any disruption to x causes the solution to tend away from the source. Since it is very likely that some stops will have fewer than the expected number of passengers and some stops will have more, it is unlikely that the time between trains will remain constant for long.

48. If the trains are spaced too close together, then each train will catch up with the one in front of it. This phenomenon will continue until there is a very large time gap between two successive trains. When this happens, the time between these two trains will grow, and a second cluster of trains will form.

For the "B branch of the Green Line," the clusters seem to contain three or four trains during rush hour. For the "D branch of the Green Line," clusters seem to contain only two trains or three trains.

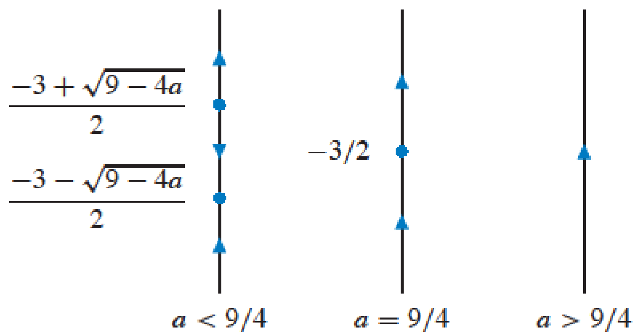
It is tempting to say that the trains should be spaced at time intervals of exactly α/β, and nothing else needs to be changed. In theory, this choice will result in equal spacing between trains, but we must remember that the equilibrium point, $x = \alpha/\beta$, is a source. Hence, anything that perturbs x will cause x to increase or decrease in an exponential fashion.

The only solution that is consistent with this model is to have the trains run to a schedule that allows for sufficient time for the loading of passengers. The trains will occasionally have to wait if they get ahead of schedule, but this plan avoids the phenomenon of one tremendously crowded train followed by two or three relatively empty ones.

2. The equilibrium points occur at solutions of $dy/dt = y^2 + 3y + a = 0$. From the quadratic formula, we have

$$y = \frac{-3 \pm \sqrt{9 - 4a}}{2}.$$

Hence, the bifurcation value of a is $9/4$. For $a < 9/4$, there are two equilibria, one source and one sink. For $a = 9/4$, there is one equilibrium which is a node, and for $a > 9/4$, there are no equilibria.



Phase lines for $a < 9/4$, $a = 9/4$, and $a > 9/4$.

3. The equilibrium points occur at solutions of $dy/dt = y^2 - ay + 1 = 0$. From the quadratic formula, we have

$$y = \frac{a \pm \sqrt{a^2 - 4}}{2}.$$

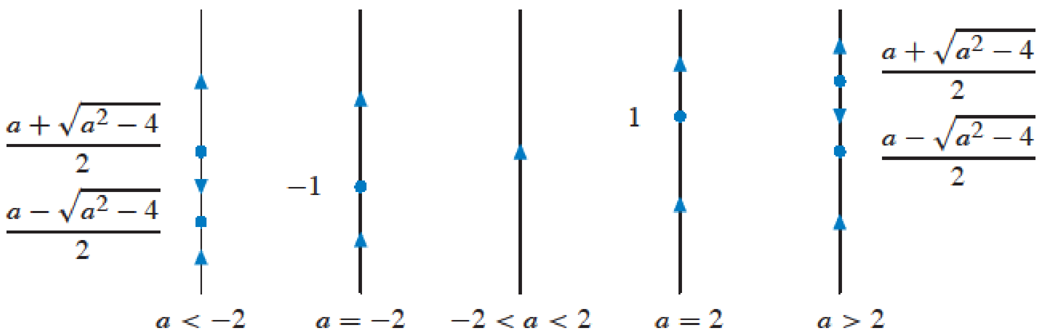
If $-2 < a < 2$, then $a^2 - 4 < 0$, and there are no equilibrium points. If $a > 2$ or $a < -2$, there are two equilibrium points. For $a = \pm 2$, there is one equilibrium point at $y = a/2$. The bifurcations occur at $a = \pm 2$.

To draw the phase lines, note that:

- For $-2 < a < 2$, $dy/dt = y^2 - ay + 1 > 0$, so the solutions are always increasing.
- For $a = 2$, $dy/dt = (y - 1)^2 \geq 0$, and $y = 1$ is a node.
- For $a = -2$, $dy/dt = (y + 1)^2 \geq 0$, and $y = -1$ is a node.
- For $a < -2$ or $a > 2$, let

$$y_1 = \frac{a - \sqrt{a^2 - 4}}{2} \quad \text{and} \quad y_2 = \frac{a + \sqrt{a^2 - 4}}{2}.$$

Then $dy/dt < 0$ if $y_1 < y < y_2$, and $dy/dt > 0$ if $y < y_1$ or $y > y_2$.



The five possible phase lines.

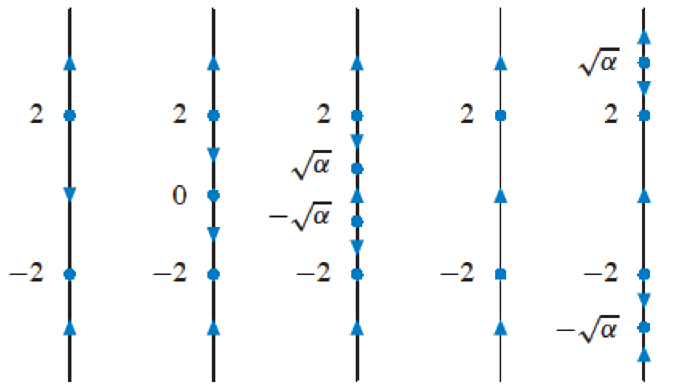
5. To find the equilibria we solve

$$(y^2-\alpha)(y^2-4)=0,$$

obtaining $y=\pm 2$ and $y=\pm\sqrt{\alpha}$ if $\alpha \geq 0$. Hence, there are two bifurcation values of α, $\alpha=0$ and $\alpha=4$.

For $\alpha<0$, there are only two equilibria. The point $y=-2$ is a sink and $y=2$ is a source. At $\alpha=0$, there are three equilibria. There is a sink at $y=-2$, a source at $y=2$, and a node at $y=0$. For $0<\alpha<4$, there are four equilibria. The point $y=-2$ is still a sink, $y=-\sqrt{\alpha}$ is a source, $y=\sqrt{\alpha}$ is a sink, and $y=2$ is still a source.

For $\alpha=4$, there are only two equilibria, $y=\pm 2$. Both are nodes. For $\alpha>4$, there are four equilibria again. The point $y=-\sqrt{\alpha}$ is a sink, $y=-2$ is now a source, $y=2$ is now a sink, and $y=\sqrt{\alpha}$ is a source.



7. We have

$$\frac{dy}{dt}=y^4+\alpha y^2=y^2(y^2+\alpha).$$

If $\alpha>0$, there is one equilibrium point at $y=0$, and $dy/dt>0$ otherwise. Hence, $y=0$ is a node.

If $\alpha<0$, there are equilibria at $y=0$ and $y=\pm\sqrt{-\alpha}$. From the sign of $y^4+\alpha y^2$, we know that $y=0$ is a node, $y=-\sqrt{-\alpha}$ is a sink, and $y=\sqrt{-\alpha}$ is a source.

The bifurcation value of α is $\alpha=0$. As α increases through 0, a sink and a source come together with the node at $y=0$, leaving only the node. For $\alpha<0$, there are three equilibria, and for $\alpha\geq 0$, there is only one equilibrium.

9. The bifurcations occur at values of α for which the graph of $\sin y + \alpha$ is tangent to the y-axis. That is, $\alpha = -1$ and $\alpha = 1$.

For $\alpha < -1$, there are no equilibria, and all solutions become unbounded in the negative direction as t increases.

If $\alpha = -1$, there are equilibrium points at $y = \pi/2 \pm 2n\pi$ for every integer n. All equilibria are nodes, and as $t \to \infty$, all other solutions decrease toward the nearest equilibrium solution below the given initial condition.

For $-1 < \alpha < 1$, there are infinitely many sinks and infinitely many sources, and they alternate along the phase line. Successive sinks differ by 2π. Similarly, successive sources are separated by 2π.

As α increases from -1 to $+1$, nearby sink and source pairs move apart. This separation continues until α is close to 1 where each source is close to the next sink with larger value of y.

At $\alpha = 1$, there are infinitely many nodes, and they are located at $y = 3\pi/2 \pm 2n\pi$ for every integer n. For $\alpha > 1$, there are no equilibria, and all solutions become unbounded in the positive direction as t increases.

11. For $\alpha = 0$, there are three equilibria. There is a sink to the left of $y = 0$, a source at $y = 0$, and a sink to the right of $y = 0$.

As α decreases, the source and sink on the right move together. A bifurcation occurs at $\alpha \approx -2$. At this bifurcation value, there is a sink to the left of $y = 0$ and a node to the right of $y = 0$. For α below this bifurcation value, there is only the sink to the left of $y = 0$.

As α increases from zero, the sink to the left of $y = 0$ and the source move together. There is a bifurcation at $\alpha \approx 2$ with a node to the left of $y = 0$ and a sink to the right of $y = 0$. For α above this bifurcation value, there is only the sink to the right of $y = 0$.

13. **(a)** Each phase line has an equilibrium point at $y = 0$. This corresponds to equations (i), (iii), and (vi). Since $y = 0$ is the only equilibrium point for $A < 0$, this only corresponds to equation (iii).

(b) The phase line corresponding to $A = 0$ is the only phase line with $y = 0$ as an equilibrium point, which corresponds to equations (ii), (iv), and (v). For the phase lines corresponding to $A < 0$, there are no equilibrium points. Only equations (iv) and (v) satisfy this property. For the phase lines corresponding to $A > 0$, note that $dy/dt < 0$ for $-\sqrt{A} < y < \sqrt{A}$. Consequently, the bifurcation diagram corresponds to equation (v).

(c) The phase line corresponding to $A = 0$ is the only phase line with $y = 0$ as an equilibrium point, which corresponds to equations (ii), (iv), and (v). For the phase lines corresponding to $A < 0$, there are no equilibrium points. Only equations (iv) and (v) satisfy this property. For the phase lines corresponding to $A > 0$, note that $dy/dt > 0$ for $-\sqrt{A} < y < \sqrt{A}$. Consequently, the bifurcation diagram corresponds to equation (iv).

(d) Each phase line has an equilibrium point at $y = 0$. This corresponds to equations (i), (iii), and (vi). The phase lines corresponding to $A > 0$ only have two nonnegative equilibrium points. Consequently, the bifurcation diagram corresponds to equation (i).

18. **(a)** For all $C \geq 0$, the equation has a source at $P = C/k$, and this is the only equilibrium point. Hence all of the phase lines are qualitatively the same, and there are no bifurcation values for C.

(b) If $P(0) > C/k$, the corresponding solution $P(t) \to \infty$ at an exponential rate as $t \to \infty$, and if $P(0) < C/k$, $P(t) \to -\infty$, passing through "extinction" ($P = 0$) after a finite time.

19. **(a)** A model of the fish population that includes fishing is

$$\frac{dP}{dt} = 2P - \frac{P^2}{50} - 3L,$$

where L is the number of licenses issued. The coefficient of 3 represents the average catch of 3 fish per year. As L is increased, the two equilibrium points for $L = 0$ (at $P = 0$ and $P = 100$) will move together. If L is sufficiently large, there are no equilibrium points. Hence we wish to pick L as large as possible so that there is still an equilibrium point present. In other words, we want the bifurcation value of L. The bifurcation value of L occurs if the equation

$$\frac{dP}{dt} = 2P - \frac{P^2}{50} - 3L = 0$$

has just one solution for P in terms of L. Using the quadratic formula, we see that there is exactly one equilibrium point if $L = 50/3$. Since this value of L is not an integer, the largest number of licenses that should be allowed is 16.

(b) If we allow the fish population to come to equilibrium then the population will be at the carrying capacity, which is $P = 100$ if $L = 0$. If we then allow 16 licenses to be issued, we expect that the population is a solution to the new model with $L = 16$ and initial population $P = 100$. The model becomes

$$\frac{dP}{dt} = 2P - \frac{P^2}{50} - 48,$$

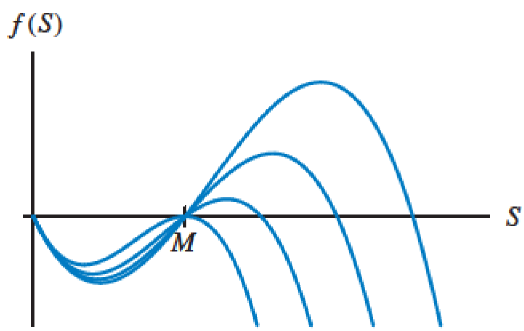
which has a source at $P = 40$ and a sink at $P = 60$.

Thus, any initial population greater than 40 when fishing begins tends to the equilibrium level $P = 60$. If the initial population of fish was less than 40 when fishing begins, then the model predicts that the population will decrease to zero in a finite amount of time.

(c) The maximum "number" of licenses is $16\frac{2}{3}$. With $L = 16\frac{2}{3}$, there is an equilibrium at $P = 50$. This equilibrium is a node, and if $P(0) > 50$, the population will approach 50 as t increases. However, it is dangerous to allow this many licenses since an unforeseen event might cause the death of a few extra fish. That event would push the number of fish below the equilibrium value of $P = 50$. In this case, $dP/dt < 0$, and the population decreases to extinction.

If, however, we restrict to $L = 16$ licenses, then there are two equilibria, a sink at $P = 60$ and source at $P = 40$. As long as $P(0) > 40$, the population will tend to 60 as t increases. In this case, we have a small margin of safety. If $P \approx 60$, then it would have to drop to less than 40 before the fish are in danger of extinction.

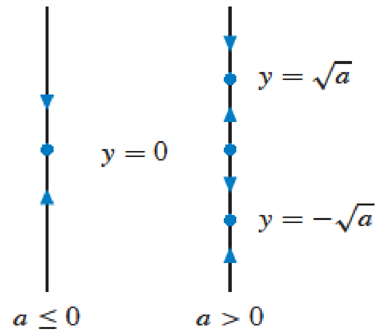
20. **(a)**



(b) The bifurcation occurs at $N = M$. The sink at $S = N$ coincides with the source at $S = M$ and becomes a node.

(c) Assuming that the population $S(t)$ is approximately N, the population adjusts to stay near the sink at $S = N$ as N slowly decreases. If $N < M$, the model is no longer consistent with the underlying assumptions.

23. **(a)** If $a \le 0$, there is a single equilibrium point at $y = 0$, and it is a sink. For $a > 0$, there are equilibrium points at $y = 0$ and $y = \pm\sqrt{a}$. The equilibrium point at $y = 0$ is a source, and the other two are sinks.



Phase lines for $dy/dt = ay - y^3$.

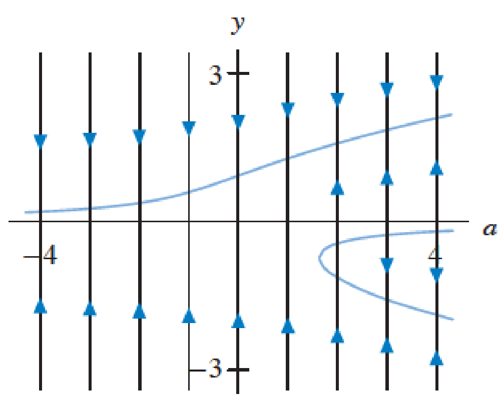
(b) Given the results in part (a), there is one bifurcation value, $a = 0$.

(c) The equilibrium points satisfy the cubic equation

$$r + ay - y^3 = 0.$$

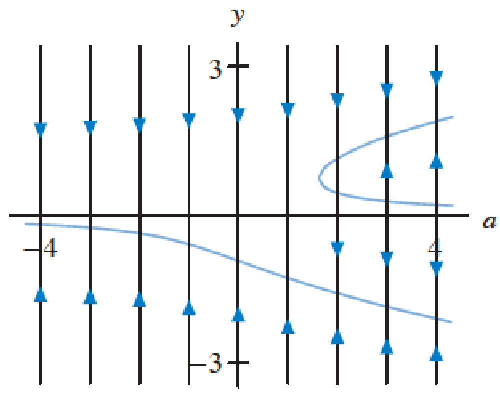
Rather than solving it explicitly, we rely on `PhaseLines`.

If $r > 0$, there is a positive bifurcation value $a = a_0$. For $a < a_0$, the phase line has one equilibrium point, a positive sink. If $a > a_0$, there are two negative equilibria in addition to the positive sink. The larger of the two negative equilibria is a source and the smaller is a sink.



The bifurcation diagram for $r = 0.8$.

(d) If $r < 0$, there is a positive bifurcation value $a = a_0$. For $a < a_0$, the phase line has one equilibrium point, a negative sink. If $a > a_0$, there are two positive equilibria in addition to the negative sink. The larger of the two positive equilibria is a sink and the smaller is a source.



The bifurcation diagram for $r = -0.8$.