1. The general solution to the associated homogeneous equation is $y_h(t) = ke^{-4t}$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha e^{-t}$. Then

$$\begin{aligned}\frac{dy_p}{dt} + 4y_p &= -\alpha e^{-t} + 4\alpha e^{-t} \\ &= 3\alpha e^{-t}.\end{aligned}$$

Consequently, we must have $3\alpha = 9$ for $y_p(t)$ to be a solution. Hence, $\alpha = 3$, and the general solution to the nonhomogeneous equation is

$$y(t) = ke^{-4t} + 3e^{-t}.$$

3. The general solution to the associated homogeneous equation is $y_h(t) = ke^{-3t}$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha \cos 2t + \beta \sin 2t$. Then

$$\begin{aligned}\frac{dy_p}{dt} + 3y_p &= -2\alpha \sin 2t + 2\beta \cos 2t + (3\alpha \cos 2t + 3\beta \sin 2t) \\ &= (3\alpha + 2\beta) \cos 2t + (3\beta - 2\alpha) \sin 2t\end{aligned}$$

Consequently, we must have

$$(3\alpha + 2\beta) \cos 2t + (3\beta - 2\alpha) \sin 2t = 4 \cos 2t$$

for $y_p(t)$ to be a solution. We must solve

$$\begin{cases} 3\alpha + 2\beta = 4 \\ 3\beta - 2\alpha = 0. \end{cases}$$

Hence, $\alpha = 12/13$ and $\beta = 8/13$. The general solution is

$$y(t) = ke^{-3t} + \tfrac{12}{13} \cos 2t + \tfrac{8}{13} \sin 2t.$$

5. The general solution to the associated homogeneous equation is $y_h(t) = ke^{3t}$. For a particular solution of the nonhomogeneous equation, we guess $y_p(t) = \alpha te^{3t}$ rather than αe^{3t} because αe^{3t} is a solution of the homogeneous equation. Then

$$\begin{aligned}\frac{dy_p}{dt} - 3y_p &= \alpha e^{3t} + 3\alpha te^{3t} - 3\alpha te^{3t} \\ &= \alpha e^{3t}.\end{aligned}$$

Consequently, we must have $\alpha = -4$ for $y_p(t)$ to be a solution. Hence, the general solution to the nonhomogeneous equation is

$$y(t) = ke^{3t} - 4te^{3t}.$$

9. The general solution of the associated homogeneous equation is $y_h(t) = ke^{-t}$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha \cos 2t + \beta \sin 2t$. Then

$$\begin{aligned} \frac{dy_p}{dt} + y_p &= -2\alpha \sin 2t + 2\beta \cos 2t + \alpha \cos 2t + \beta \sin 2t \\ &= (\alpha + 2\beta) \cos 2t + (-2\alpha + \beta) \sin 2t. \end{aligned}$$

Consequently, we must have

$$(\alpha + 2\beta) \cos 2t + (-2\alpha + \beta) \sin 2t = \cos 2t$$

for $y_p(t)$ to be a solution. We must solve

$$\begin{cases} \alpha + 2\beta = 1 \\ -2\alpha + \beta = 0. \end{cases}$$

Hence, $\alpha = 1/5$ and $\beta = 2/5$. The general solution to the differential equation is

$$y(t) = ke^{-t} + \tfrac{1}{5} \cos 2t + \tfrac{2}{5} \sin 2t.$$

To find the solution of the given initial-value problem, we evaluate the general solution at $t = 0$ and obtain

$$y(0) = k + \tfrac{1}{5}.$$

Since the initial condition is $y(0) = 5$, we see that $k = 24/5$. The desired solution is

$$y(t) = \tfrac{24}{5} e^{-t} + \tfrac{1}{5} \cos 2t + \tfrac{2}{5} \sin 2t.$$

11. The general solution to the associated homogeneous equation is $y_h(t) = ke^{2t}$. For a particular solution of the nonhomogeneous equation, we guess $y_p(t) = \alpha t e^{2t}$ rather than αe^{2t} because αe^{2t} is a solution of the homogeneous equation. Then

$$\begin{aligned} \frac{dy_p}{dt} - 2y_p &= \alpha e^{2t} + 2\alpha t e^{2t} - 2\alpha t e^{2t} \\ &= \alpha e^{2t}. \end{aligned}$$

Consequently, we must have $\alpha = 7$ for $y_p(t)$ to be a solution. Hence, the general solution to the nonhomogeneous equation is

$$y(t) = ke^{2t} + 7te^{2t}.$$

Note that $y(0) = k = 3$, so the solution to the initial-value problem is

$$y(t) = 3e^{2t} + 7te^{2t} = (3 + 7t)e^{2t}.$$

17. **(a)** We compute

$$\frac{dy_1}{dt} = \frac{1}{(1-t)^2} = (y_1(t))^2$$

to see that $y_1(t)$ is a solution.

(b) We compute

$$\frac{dy_2}{dt} = 2\frac{1}{(1-t)^2} \neq (y_2(t))^2$$

to see that $y_2(t)$ is not a solution.

(c) The equation $dy/dt = y^2$ is not linear. It contains y^2.

20. If $y_p(t) = at^2 + bt + c$, then

$$\begin{aligned}\frac{dy_p}{dt} + 2y_p &= 2at + b + 2at^2 + 2bt + 2c \\ &= 2at^2 + (2a + 2b)t + (b + 2c).\end{aligned}$$

Then $y_p(t)$ is a solution if this quadratic is equal to $3t^2 + 2t - 1$. In other words, $y_p(t)$ is a solution if

$$\begin{cases} 2a = 3 \\ 2a + 2b = 2 \\ b + 2c = -1. \end{cases}$$

From the first equation, we have $a = 3/2$. Then from the second equation, we have $b = -1/2$. Finally, from the third equation, we have $c = -1/4$. The function

$$y_p(t) = \tfrac{3}{2}t^2 - \tfrac{1}{2}t - \tfrac{1}{4}$$

is a solution of the differential equation.

21. To find the general solution, we use the technique suggested in Exercise 19. We calculate two particular solutions—one for the right-hand side $t^2 + 2t + 1$ and one for the right-hand side e^{4t}.

With the right-hand side $t^2 + 2t + 1$, we guess a solution of the form

$$y_{p_1}(t) = at^2 + bt + c.$$

Then

$$\begin{aligned}\frac{dy_{p_1}}{dt} + 2y_{p_1} &= 2at + b + 2(at^2 + bt + c)\\ &= 2at^2 + (2a + 2b)t + (b + 2c).\end{aligned}$$

Then y_{p_1} is a solution if

$$\begin{cases} 2a = 1 \\ 2a + 2b = 2 \\ b + 2c = 1. \end{cases}$$

We get $a = 1/2$, $b = 1/2$, and $c = 1/4$.

With the right-hand side e^{4t}, we guess a solution of the form

$$y_{p_2}(t) = \alpha e^{4t}.$$

Then

$$\frac{dy_{p_2}}{dt} + 2y_{p_2} = 4\alpha e^{4t} + 2\alpha e^{4t} = 6\alpha e^{4t},$$

and y_{p_2} is a solution if $\alpha = 1/6$.

The general solution of the associated homogeneous equation is $y_h(t) = ke^{-2t}$, so the general solution of the original equation is

$$ke^{-2t} + \tfrac{1}{2}t^2 + \tfrac{1}{2}t + \tfrac{1}{4} + \tfrac{1}{6}e^{4t}.$$

To find the solution that satisfies the initial condition $y(0) = 0$, we evaluate the general solution at $t = 0$ and obtain

$$k + \tfrac{1}{4} + \tfrac{1}{6} = 0.$$

Hence, $k = -5/12$.

25. Since the general solution of the associated homogeneous equation is $y_h(t) = ke^{-2t}$ and since these $y_h(t) \to 0$ as $t \to \infty$, we only have to determine the long-term behavior of one solution to the nonhomogeneous equation. However, that is easier said than done.

Consider the slopes in the slope field for the equation. We rewrite the equation as

$$\frac{dy}{dt} = -2y + b(t).$$

Using the fact that $b(t) < 2$ for all t, we observe that $dy/dt < 0$ if $y > 1$ and, as y increases beyond $y = 1$, the slopes become more negative. Similarly, using the fact that $b(t) > -1$ for all t, we observe that $dy/dt > 0$ if $y < -1/2$ and, as y decreases below $y = -1/2$, the slopes become more positive. Thus, the graphs of all solutions must approach the strip $-1/2 \leq y \leq 1$ in the ty-plane as t increases. More precise information about the long-term behavior of solutions is difficult to obtain without specific knowledge of $b(t)$.

30. Let $M(t)$ be the amount of money left at time t. Then, we have the initial condition $M(0) = \$70{,}000$. Money is being added to the account at a rate of 1.5% and removed from the account at a rate of \$30,000 per year, so

$$\frac{dM}{dt} = 0.015M - 30{,}000.$$

To find the general solution, we first compute the general solution of the associated homogeneous equation. It is $M_h(t) = ke^{0.015t}$.

To find a particular solution of the nonhomogeneous equation, we observe that the equation is autonomous, and we calculate its equilibrium solution. It is $M(t) = 30{,}000/.015 = \$2{,}000{,}000$ for all t. (This equilibrium solution is what we would have calculated if we had guessed a constant.)

Therefore we have

$$M(t) = 2{,}000{,}000 + ke^{0.015t}.$$

Using the initial condition $M(0) = 70{,}000$, we have

$$2{,}000{,}000 + k = 70{,}000,$$

so $k = -1{,}930{,}000$ and

$$M(t) = 2{,}000{,}000 - 1{,}930{,}000e^{0.015t}.$$

Solving for the value of t when $M(t) = 0$, we have

$$2{,}000{,}000 - 1{,}930{,}000e^{0.015t} = 0,$$

which is equivalent to

$$e^{0.015t} = \frac{2{,}000{,}000}{1{,}930{,}000}.$$

In other words,

$$0.015t = \ln(1.03627),$$

which yields $t \approx 2.375$ years.

1. We rewrite the equation in the form

$$\frac{dy}{dt} + \frac{y}{t} = 2$$

and note that the integrating factor is

$$\mu(t) = e^{\int (1/t)\,dt} = e^{\ln t} = t.$$

Multiplying both sides by $\mu(t)$, we obtain

$$t\frac{dy}{dt} + y = 2t.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d(ty)}{dt} = 2t,$$

and integrating both sides with respect to t, we obtain

$$ty = t^2 + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = \frac{1}{t}(t^2 + c) = t + \frac{c}{t}.$$

5. Note that the integrating factor is

$$\mu(t) = e^{\int (-2t/(1+t^2))\,dt} = e^{-\ln(1+t^2)} = \left(e^{\ln(1+t^2)}\right)^{-1} = \frac{1}{1+t^2}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$\frac{1}{1+t^2}\frac{dy}{dt} - \frac{2t}{(1+t^2)^2}\,y = \frac{3}{1+t^2}.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d}{dt}\left(\frac{y}{1+t^2}\right) = \frac{3}{1+t^2}.$$

Integrating both sides with respect to t, we obtain

$$\frac{y}{1+t^2} = 3\arctan(t) + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = (1+t^2)(3\arctan(t) + c).$$

7. We rewrite the equation in the form

$$\frac{dy}{dt} + \frac{y}{1+t} = 2$$

and note that the integrating factor is

$$\mu(t) = e^{\int (1/(1+t))\,dt} = e^{\ln(1+t)} = 1 + t.$$

Multiplying both sides by $\mu(t)$, we obtain

$$(1+t)\frac{dy}{dt} + y = 2(1+t).$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d((1+t)y)}{dt} = 2(1+t),$$

and integrating both sides with respect to t, we obtain

$$(1+t)y = 2t + t^2 + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = \frac{t^2 + 2t + c}{1+t}.$$

To find the solution that satisfies the initial condition $y(0) = 3$, we evaluate the general solution at $t = 0$ and obtain

$$c = 3.$$

The desired solution is

$$y(t) = \frac{t^2 + 2t + 3}{1+t}.$$

11. Note that the integrating factor is

$$\mu(t) = e^{\int -(2/t)\,dt} = e^{-2\int (1/t)\,dt} = e^{-2\ln t} = e^{\ln(t^{-2})} = \frac{1}{t^2}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$\frac{1}{t^2}\frac{dy}{dt} - \frac{2y}{t^3} = 2.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d}{dt}\left(\frac{y}{t^2}\right) = 2,$$

and integrating both sides with respect to t, we obtain

$$\frac{y}{t^2} = 2t + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = 2t^3 + ct^2.$$

To find the solution that satisfies the initial condition $y(-2) = 4$, we evaluate the general solution at $t = -2$ and obtain

$$-16 + 4c = 4.$$

Hence, $c = 5$, and the desired solution is

$$y(t) = 2t^3 + 5t^2.$$

13. We rewrite the equation in the form

$$\frac{dy}{dt} - (\sin t)y = 4$$

and note that the integrating factor is

$$\mu(t) = e^{\int (-\sin t)\, dt} = e^{\cos t}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$e^{\cos t}\frac{dy}{dt} - e^{\cos t}(\sin t)y = 4e^{\cos t}.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d(e^{\cos t}y)}{dt} = 4e^{\cos t},$$

and integrating both sides with respect to t, we obtain

$$e^{\cos t}y = \int 4e^{\cos t}\, dt.$$

Since the integral on the right-hand side is impossible to express using elementary functions, we write the general solution as

$$y(t) = 4e^{-\cos t}\int e^{\cos t}\, dt.$$

17. We rewrite the equation in the form

$$\frac{dy}{dt} + e^{-t^2} y = \cos t$$

and note that the integrating factor is

$$\mu(t) = e^{\int e^{-t^2}\,dt}.$$

This integral is impossible to express in terms of elementary functions. Multiplying both sides by $\mu(t)$, we obtain

$$\left(e^{\int e^{-t^2}\,dt}\right) \frac{dy}{dt} + \left(e^{\int e^{-t^2}\,dt}\right) e^{-t^2} y = \left(e^{\int e^{-t^2}\,dt}\right) \cos t.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d\left(\left(e^{\int e^{-t^2}\,dt}\right) y\right)}{dt} = \left(e^{\int e^{-t^2}\,dt}\right) \cos t,$$

and integrating both sides with respect to t, we obtain

$$\left(e^{\int e^{-t^2}\,dt}\right) y = \int \left(e^{\int e^{-t^2}\,dt}\right) \cos t\,dt.$$

These integrals are also impossible to express in terms of elementary functions, so we write the general solution in the form

$$y(t) = \left(e^{-\int e^{-t^2}\,dt}\right) \int \left(e^{\int e^{-t^2}\,dt}\right) \cos t\,dt.$$

21. **(a)** The integrating factor is

$$\mu(t) = e^{0.4t}.$$

Multiplying both sides of the differential equation by $\mu(t)$ and collecting terms, we obtain

$$\frac{d(e^{0.4t}v)}{dt} = 3e^{0.4t}\cos 2t.$$

Integrating both sides with respect to t yields

$$e^{0.4t}v = \int 3e^{0.4t}\cos 2t\,dt.$$

To calculate the integral on the right-hand side, we must integrate by parts twice.

For the first integration, we pick $u_1(t) = \cos 2t$ and $v_1(t) = e^{0.4t}$. Using the fact that $0.4 = 2/5$, we get

$$\int e^{0.4t}\cos 2t\,dt = \tfrac{5}{2}\,e^{0.4t}\cos 2t + 5\int e^{0.4t}\sin 2t\,dt.$$

For the second integration, we pick $u_2(t) = \sin 2t$ and $v_2(t) = e^{0.4t}$. We get

$$\int e^{0.4t}\sin 2t\,dt = \tfrac{5}{2}e^{0.4t}\sin 2t - 5\int e^{0.4t}\cos 2t\,dt.$$

Combining these results yields

$$\int e^{0.4t}\cos 2t\,dt = \tfrac{5}{2}\,e^{0.4t}\cos 2t + \tfrac{25}{2}\,e^{0.4t}\sin 2t - 25\int e^{0.4t}\cos 2t\,dt.$$

Solving for $\int e^{0.4t}\cos 2t\,dt$, we have

$$\int e^{0.4t}\cos 2t\,dt = \frac{5\,e^{0.4t}\cos 2t + 25\,e^{0.4t}\sin 2t}{52}.$$

To obtain the general solution, we multiply this integral by 3, add the constant of integration, and solve for v. We obtain the general solution

$$v(t) = ke^{-0.4t} + \tfrac{15}{52}\cos 2t + \tfrac{75}{52}\sin 2t.$$

(b) The solution of the associated homogeneous equation is

$$v_h(t) = e^{-0.4t}.$$

We guess

$$v_p(t) = \alpha \cos 2t + \beta \sin 2t$$

for the a solution to the nonhomogeneous equation and solve for α and β. Substituting this guess into the differential equation, we obtain

$$-2\alpha \sin 2t + 2\beta \cos 2t + 0.4\alpha \cos 2t + 0.4\beta \sin 2t = 3 \cos 2t.$$

Collecting sine and cosine terms, we get the system of equations

$$\begin{cases} -2\alpha + 0.4\beta = 0 \\ 0.4\alpha + 2\beta = 3. \end{cases}$$

Using the fact that $0.4 = 2/5$, we solve this system of equations and obtain

$$\alpha = \tfrac{15}{52} \quad \text{and} \quad \beta = \tfrac{75}{52}.$$

The general solution of the original nonhomogeneous equation is

$$v(t) = ke^{-0.4t} + \tfrac{15}{52} \cos 2t + \tfrac{75}{52} \sin 2t.$$

Both methods require quite a bit of computation. If we use an integrating factor, we must do a complicated integral, and if we use the guessing technique, we have to be careful with our algebra.

25. We will use the term "parts" as shorthand for the product of parts per billion of dioxin and the volume of water in the tank. Basically this product represents the total amount of dioxin in the tank. The tank initially contains 200 gallons at a concentration of 2 parts per billion, which results in 400 parts of dioxin.

Let $y(t)$ be the amount of dioxin in the tank at time t. Since water with 4 parts per billion of dioxin flows in at the rate of 5 gallons per minute, 20 parts of dioxin enter the tank each minute. Also, the volume of water in the tank at time t is $200 + 2t$, so the concentration of dioxin in the tank is $y/(200 + 2t)$. Since well-mixed water leaves the tank at the rate of 2 gallons per minute, the differential equation that represents the change in the amount of dioxin in the tank is

$$\frac{dy}{dt} = 20 - 2\left(\frac{y}{200 + 2t}\right),$$

which can be simplified and rewritten as

$$\frac{dy}{dt} + \left(\frac{1}{100 + t}\right) y = 20.$$

The integrating factor is

$$\mu(t) = e^{\int (1/(100+t))\, dt} = e^{\ln(100+t)} = 100 + t.$$

Multiplying both sides by $\mu(t)$, we obtain

$$(100 + t)\frac{dy}{dt} + y = 20(100 + t),$$

which is equivalent to

$$\frac{d((100 + t)y)}{dt} = 20(100 + t)$$

by the Product Rule. Integrating both sides with respect to t, we obtain

$$(100 + t)y = 2000t + 10t^2 + c.$$

Since $y(0) = 400$, we see that $c = 40,000$. Therefore,

$$y(t) = \frac{10t^2 + 2000t + 40,000}{t + 100}.$$

The tank fills up at $t = 100$, and $y(100) = 1,700$. To express our answer in terms of concentration, we calculate $y(100)/400 = 4.25$ parts per billion.