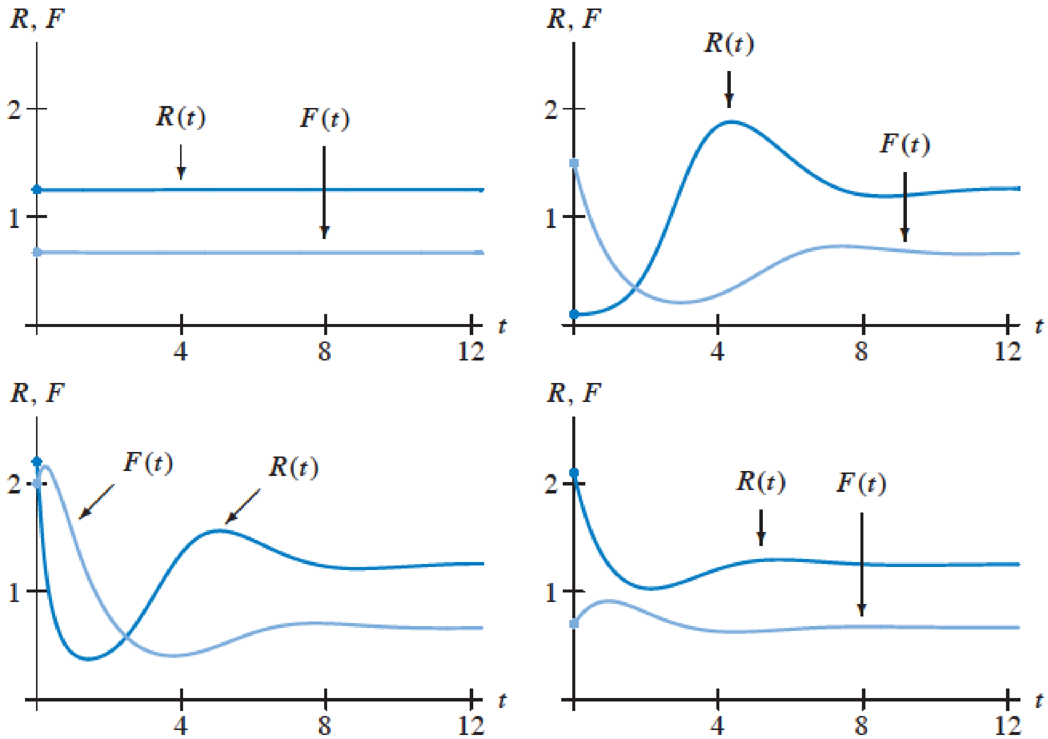
7. The population starts with a relatively large rabbit (R) and a relatively small fox (F) population. The rabbit population grows, then the fox population grows while the rabbit population decreases. Next the fox population decreases until both populations are close to zero. Then the rabbit population grows again and the cycle starts over. Each repeat of the cycle is less dramatic (smaller total oscillation) and both populations oscillate toward an equilibrium which is approximately $(R, F) = (1/2, 3/2)$.

8. **(a)**



(b) Each of the solutions tends to the equilibrium point at $(R, F) = (5/4, 2/3)$. The populations of both species tend to a limit and the species coexist. For curve B, note that the F-population initially decreases while R increases. Eventually F bottoms out and begins to rise. Then R peaks and begins to fall. Then both populations tend to the limit.

9. By hunting, the number of prey decreases α units per unit of time. Therefore, the rate of change dR/dt of the number of prey has the term $-\alpha$. Only the equation for dR/dt needs modification.

(i) $dR/dt = 2R - 1.2RF - \alpha$

(ii) $dR/dt = R(2 - R) - 1.2RF - \alpha$

10. Hunting decreases the number of predators by an amount proportional to the number of predators alive (that is, by a term of the form $-kF$), so we have $dF/dt = -F + 0.9RF - kF$ in each case.

11. Since the second food source is unlimited, if $R = 0$ and k is the growth parameter for the predator population, F obeys an exponential growth model, $dF/dt = kF$. The only change we have to make is in the rate of F, dF/dt. For both (i) and (ii), $dF/dt = kF + 0.9RF$.

12. In the absence of prey, the predators would obey a logistic growth law. So we could modify both systems by adding a term of the form $-kF/N$, where k is the growth-rate parameter and N is the carrying capacity of predators. That is, we have $dF/dt = kF(1 - F/N) + 0.9RF$.

13. If $R - 5F > 0$, the number of predators increases and, if $R - 5F < 0$, the number of predators decreases. Since the condition on prey is same, we modify only the predator part of the system. the modified rate of change of the predator population is

$$\frac{dF}{dt} = -F + 0.9RF + k(R - 5F)$$

where $k > 0$ is the immigration parameter for the predator population.

14. In both cases the rate of change of population of prey decreases by a factor of kF. Hence we have

(i) $dR/dt = 2R - 1.2RF - kF$

(ii) $dR/dt = 2R - R^2 - 1.2RF - kF$

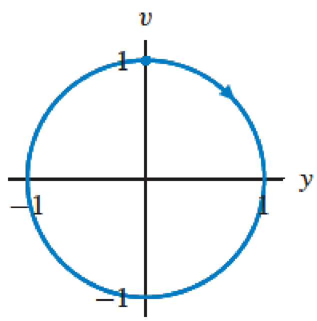
15. Suppose $y = 1$. If we can find a value of x such that $dy/dt = 0$, then for this x and $y = 1$ the predator population is constant. (This point may not be an equilibrium point because we do not know if $dx/dt = 0$.) The required value of x is $x = 0.05$ in system (i) and $x = 20$ in system (ii). Survival for one unit of predators requires 0.05 units of prey in (i) and 20 units of prey in (ii). Therefore, (i) is a system of inefficient predators and (ii) is a system of efficient predators.

19. **(a)** Substituting $y(t) = \sin t$ into the left-hand side of the differential equation gives

$$\frac{d^2y}{dt^2} + y = \frac{d^2(\sin t)}{dt^2} + \sin t$$
$$= -\sin t + \sin t$$
$$= 0,$$

so the left-hand side equals the right-hand side for all t.

(b)



(c) These two solutions trace the same curve in the yv-plane—the unit circle.

(d) The difference in the two solution curves is in how they are parameterized. The solution in this problem is at $(0, 1)$ at time $t = 0$ and hence it lags behind the solution in the section by $\pi/2$. This information cannot be observed solely by looking at the solution curve in the phase plane.

22. **(a)** First, we need to determine the spring constant k. Using Hooke's law, we have 4 lbs $= k \cdot 4$ in. Thus, $k = 1$ lbs/in $= 12$ lbs/ft. We will measure distance in feet since the mass is extended 1 foot.

To determine the mass of a 4 lb object, we use the fact that the force due to gravity is mg where $g = 32$ ft/sec^2. Thus, $m = 4/32 = 1/8$.

Using the model

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0,$$

for the undamped harmonic oscillator, we obtain

$$\frac{d^2y}{dt^2} + 96y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

as our initial-value problem.

(b) From Exercise 20 we know that $y(t) = \cos \beta t$ is a solution to the differential equation for the simple harmonic oscillator, where $\beta = \sqrt{k/m}$. Since $y(t) = \cos \sqrt{96}\,t$ satisfies both our differential equation and our initial conditions, it is the solution to the initial-value problem.

23. An extra firm mattress does not deform when you lay on it. This means that it takes a great deal of force to compress the springs so the spring constant must be large.

25. Suppose $\alpha > 0$ is the reaction rate constant for A+B $\to$ C. The reaction rate is αab at time t, and after the reaction, a and b decrease by αab. We therefore obtain the system

$$\frac{da}{dt} = -\alpha ab$$
$$\frac{db}{dt} = -\alpha ab.$$

26. Measure the amount of C produced during the short time interval from $t = 0$ to $t = \Delta t$. The amount is given by $a(0) - a(\Delta t)$ since one molecule of A yields one molecule of C. Now

$$\frac{a(0) - a(\Delta t)}{\Delta t} \approx -a'(0) = \alpha a(0)b(0).$$

Since we know $a(0)$, $a(\Delta t)$, $b(0)$, and Δt, we can therefore solve for α.

27. Suppose k_1 and k_2 are the rates of increase of A and B respectively. Since A and B are added to the solution at constant rates, k_1 and k_2 are added to da/dt and db/dt respectively. The system becomes

$$\frac{da}{dt} = k_1 - \alpha ab$$
$$\frac{db}{dt} = k_2 - \alpha ab.$$

28. The chance that two A molecules are close is proportional to a^2. Hence, the new system is

$$\frac{da}{dt} = k_1 - \alpha ab - \gamma a^2$$
$$\frac{db}{dt} = k_2 - \alpha ab,$$

where γ is a parameter that measures the rate at which A combines to make D.

29. Suppose γ is the reaction-rate coefficient for the reaction B + B $\to$ A. By the reaction, two B's react with each other to create one A. In other words, B decreases at the rate γb^2 and A increases at the rate $\gamma b^2/2$. The resulting system of the differential equations is

$$\frac{da}{dt} = k_1 - \alpha ab + \frac{\gamma b^2}{2}$$
$$\frac{db}{dt} = k_2 - \alpha ab - \gamma b^2.$$

30. The chance that two B's and an A molecule are close is proportional to ab^2, so

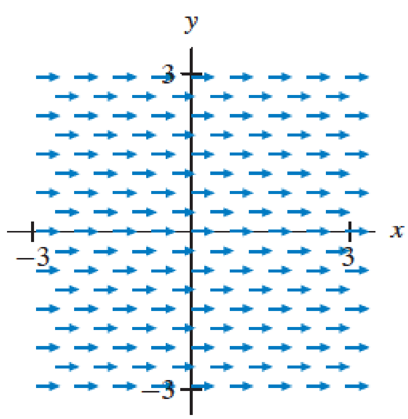
$$\frac{da}{dt} = k_1 - \alpha ab - \gamma ab^2$$
$$\frac{db}{dt} = k_2 - \alpha ab - 2\gamma ab^2,$$

where γ is the reaction-rate parameter for the reaction that produces D from two B's and an A.

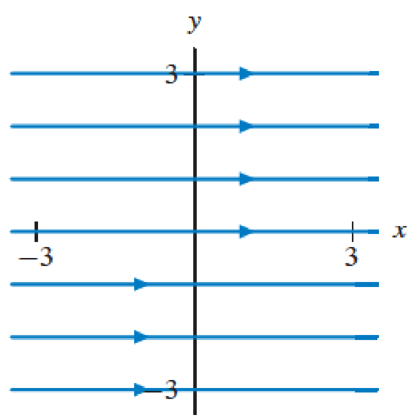
1. **(a)** $\mathbf{V}(x, y) = (1, 0)$

(b) See part (c).

(c)



(d)

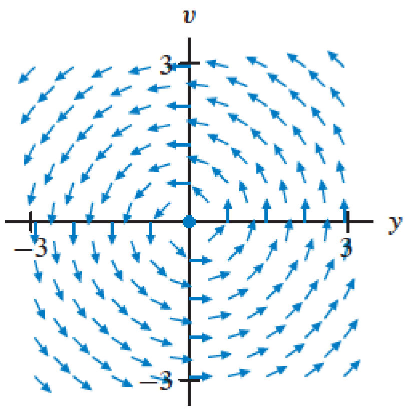


(e) As t increases, solutions move along horizontal lines toward the right.

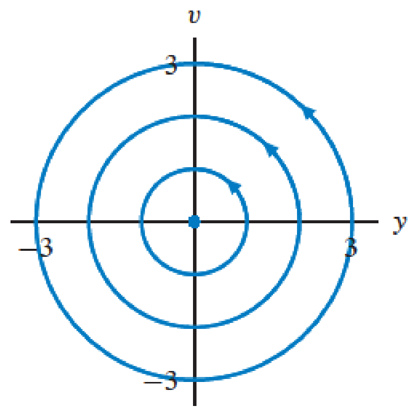
3. **(a)** $\mathbf{V}(y, v) = (-v, y)$

(b) See part (c).

(c)



(d)

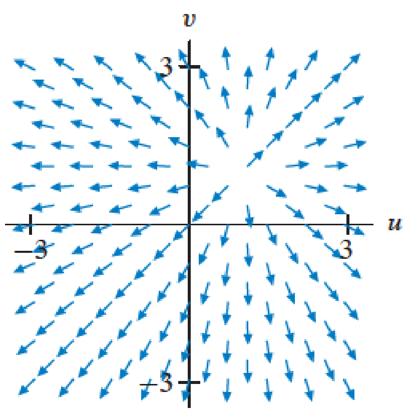


(e) As t increases, solutions move on circles around $(0, 0)$ in the counter-clockwise direction.

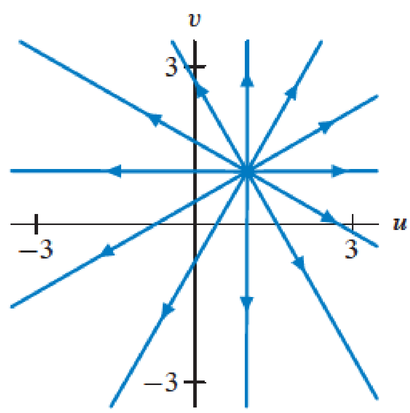
4. **(a)** $\mathbf{V}(u, v) = (u - 1, v - 1)$

(b) See part (c).

(c)



(d)

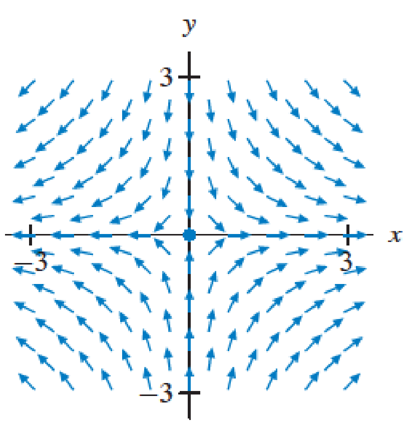


(e) As t increases, solutions move away from the equilibrium point at $(1, 1)$.

5. **(a)** $\mathbf{V}(x, y) = (x, -y)$

(b) See part (c).

(c)



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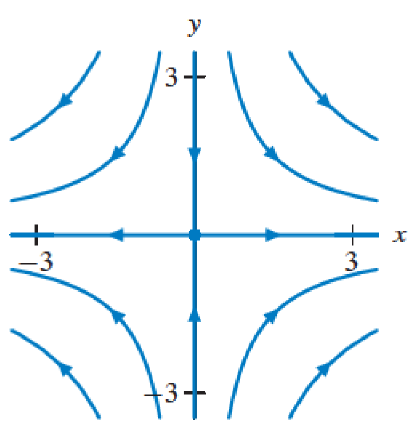


(e) As t increases, solutions move toward the x-axis in the y-direction and away from the y-axis in the x-direction.

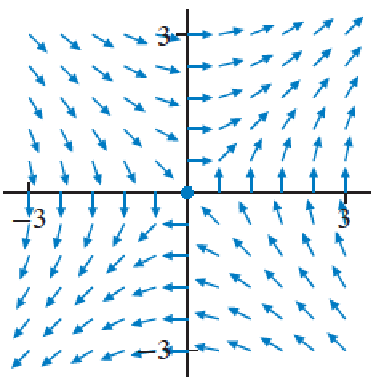
7. **(a)** Let $v = dy/dt$. Then

$$\frac{dv}{dt} = \frac{d^2y}{dt^2} = y.$$

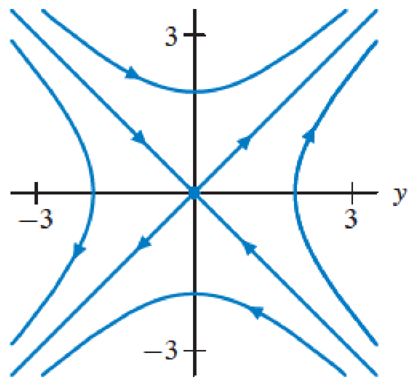
Thus the associated vector field is $\mathbf{V}(y, v) = (v, y)$.

(b) See part (c).

(c)



(d)



(e) As t increases, solutions in the 2nd and 4th quadrants move toward the origin and away from the line $y = -v$. Solutions in the 1st and 3rd quadrants move away from the origin and toward the line $y = v$.

11. **(a)** There are equilibrium points at $(\pm 1, 0)$, so only systems (ii) and (vii) are possible. Since the direction field points toward the x-axis if $y \neq 0$, the equation $dy/dt = y$ does not match this field. Therefore, system (vii) is the system that generated this direction field.

(b) The origin is the only equilibrium point, so the possible systems are (iii), (iv), (v), and (viii). The direction field is not tangent to the y-axis, so it does not match either system (iv) or (v). Vectors point toward the origin on the line $y = x$, so $dy/dt = dx/dt$ if $y = x$. This condition is not satisfied by system (iii). Consequently, this direction field corresponds to system (viii).

(c) The origin is the only equilibrium point, so the possible systems are (iii), (iv), (v), and (viii). Vectors point directly away from the origin on the y-axis, so this direction field does not correspond to systems (iii) and (viii). Along the line $y = x$, the vectors are more vertical than horizontal. Therefore, this direction field corresponds to system (v) rather than system (iv).

(d) The only equilibrium point is $(1, 0)$, so the direction field must correspond to system (vi).

12. The equilibrium solutions are those solutions for which $dR/dt = 0$ and $dF/dt = 0$ simultaneously. To find the equilibrium points, we must solve the system of equations

$$\begin{cases} 2R\left(1 - \dfrac{R}{2}\right) - 1.2RF = 0 \\ -F + 0.9RF = 0. \end{cases}$$

The second equation is satisfied if $F = 0$ or if $R = 10/9$, and we consider each case independently. If $F = 0$, then the first equation is satisfied if and only if $R = 0$ or $R = 2$. Thus two equilibrium solutions are $(R, F) = (0, 0)$ and $(R, F) = (2, 0)$.

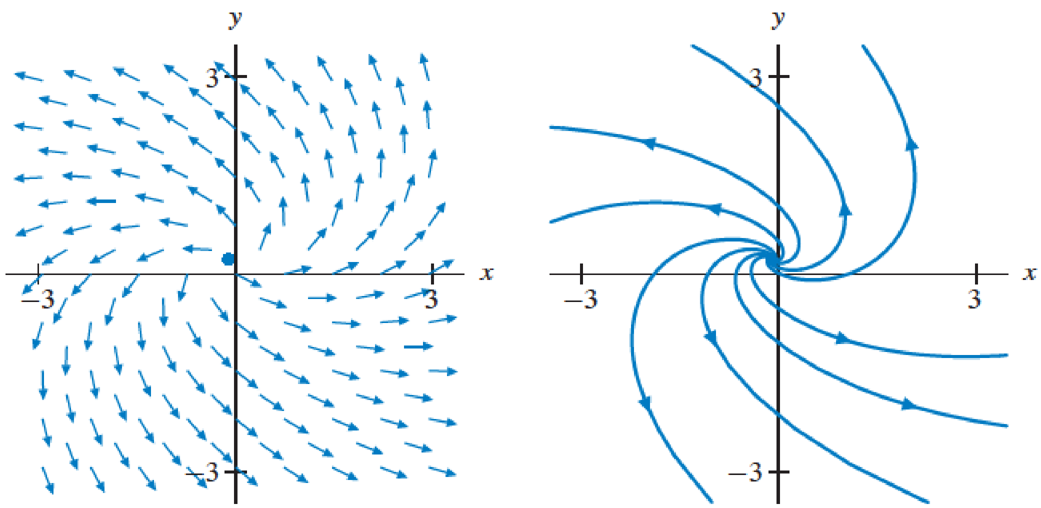
If $R = 10/9$, we substitute this value into the first equation and obtain $F = 20/27$.

13. **(a)** To find the equilibrium points, we solve the system of equations

$$\begin{cases} 4x - 7y + 2 = 0 \\ 3x + 6y - 1 = 0. \end{cases}$$

These simultaneous equations have one solution, $(x, y) = (-1/9, 2/9)$.

(b)

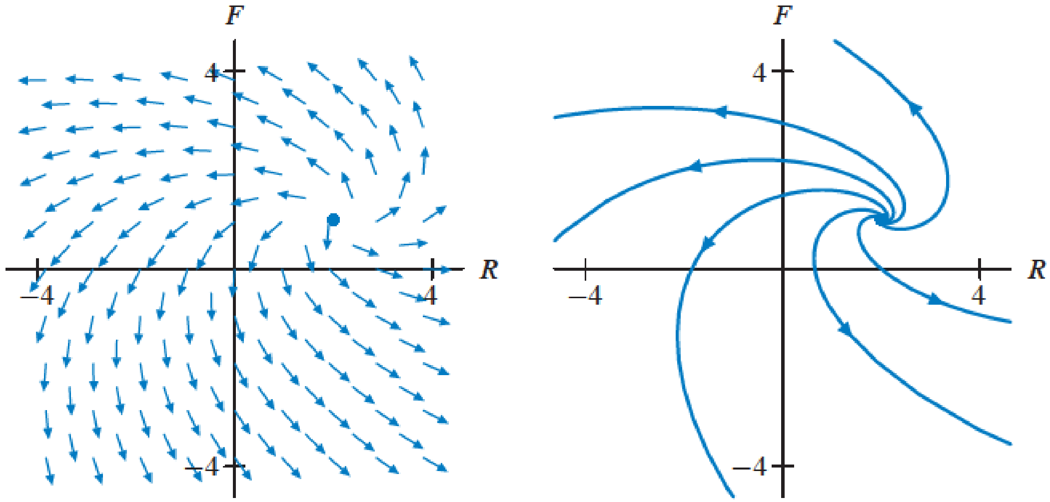


(c) As t increases, typical solutions spiral away from the origin in the counter-clockwise direction.

14. **(a)** To find the equilibrium points, we solve the system of equations

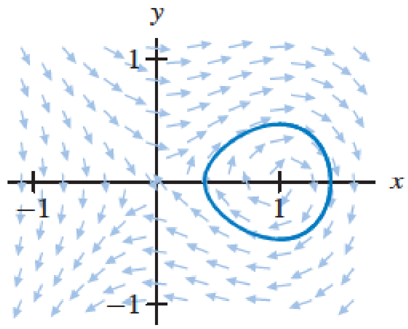
$$\begin{cases} 4R - 7F - 1 = 0 \\ 3R + 6F - 12 = 0. \end{cases}$$

These simultaneous equations have one solution, $(R, F) = (2, 1)$.

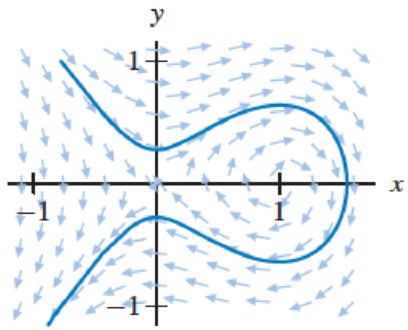


(b) As t increases, typical solutions spiral away from the equilibrium point at (2, 1)

21. **(a)** The $x(t)$- and $y(t)$-graphs are periodic, so they correspond to a solution curve that returns to its initial condition in the phase plane. In other words, its solution curve is a closed curve. Since the amplitude of the oscillation of $x(t)$ is relatively large, these graphs must correspond to the outermost closed solution curve.

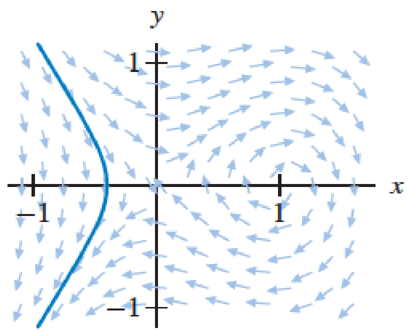


(b) The graphs are not periodic, so they cannot correspond to the two closed solution curves in the phase portrait. Both graphs cross the t-axis. The value of $x(t)$ is initially negative, then becomes positive and reaches a maximum, and finally becomes negative again. Therefore, the corresponding solution curve is the one that starts in the second quadrant, then travels through the first and fourth quadrants, and finally enters the third quadrant.

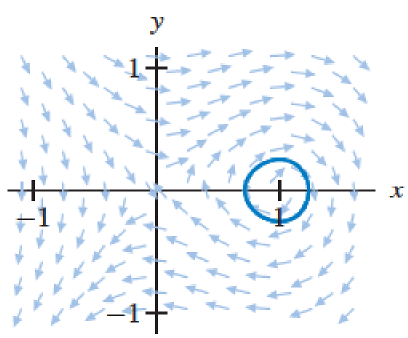


(c) The graphs are not periodic, so they cannot correspond to the two closed solution curves in the phase portrait. Only one graph crosses the t-axis. The other graph remains negative for all time. Note that the two graphs cross.

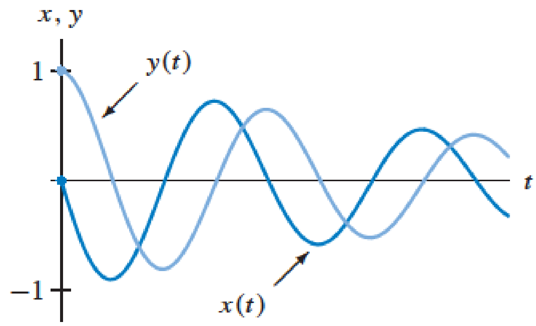
The corresponding solution curve is the one that starts in the second quadrant and crosses the x-axis and the line $y = x$ as it moves through the third quadrant.



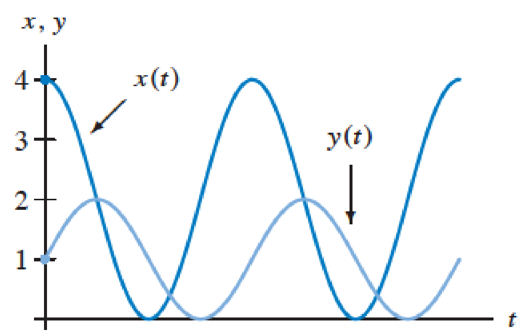
(d) The $x(t)$- and $y(t)$-graphs are periodic, so they correspond to a solution curve that returns to its initial condition in the phase plane. In other words, its solution curve is a closed curve. Since the amplitude of the oscillation of $x(t)$ is relatively small, these graphs must correspond to the intermost closed solution curve.



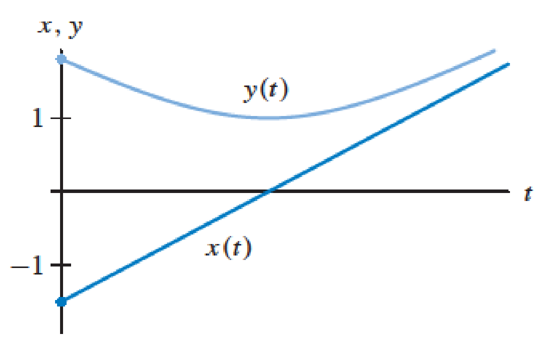
23. Since the solution curve spirals into the origin, the corresponding $x(t)$- and $y(t)$-graphs must oscillate about the t-axis with the decreasing amplitudes.



24. Since the solution curve is an ellipse that is centered at $(2, 1)$, the $x(t)$- and $y(t)$-graphs are periodic. They oscillate about the lines $x = 2$ and $y = 1$.



25. The $x(t)$-graph satisfies $-2 < x(0) < -1$ and increases as t increases. The $y(t)$-graph satisfies $1 < y(0) < 2$. Initially it decreases until it reaches its minimum value of $y = 1$ when $x = 0$. Then it increases as t increases.



26. The $x(t)$-graph starts with a small positive value and increases as t increases. The $y(t)$-graph starts at approximately 1.6 and decreases as t increases. However, $y(t)$ remains positive for all t.