1. The system of differential equations is

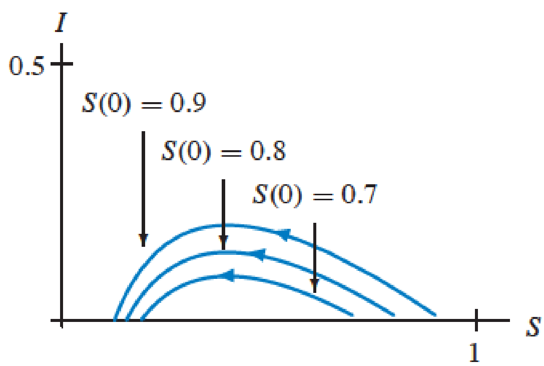
$$\frac{dS}{dt} = -\alpha SI$$
$$\frac{dI}{dt} = \alpha SI - \beta I$$
$$\frac{dR}{dt} = \beta I.$$

Note that

$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = -\alpha SI + (\alpha SI - \beta I) + \beta I = 0.$$

Hence, the sum $S(t) + I(t) + R(t)$ is constant for all t. Since the model assumes that the total population is divided into these three groups at $t = 0$, $S(0) + I(0) + R(0) = 1$. Therefore, $S(t) + I(t) + R(t) = 1$ for all t.

2. **(a)**



As $S(0)$ decreases, the maximum of $I(t)$ decreases, that is, the maximum number of infecteds decreases as the initial proportion of the susceptible population decreases. Furthermore, as $S(0)$ decreases, the limit of $S(t)$ as $t \to \infty$ increases. Consequently, the fraction of the population that contracts the disease during the epidemic decreases as the initial proportion of the susceptible population decreases.

(b) If $\alpha = 0.25$ and $\beta = 0.1$, the threshold value of the model is $\beta/\alpha = 0.1/0.25 = 0.4$. If $S(0) < 0.4$, then $dI/dt < 0$ for all $t > 0$. In other words, any influx of infecteds will decrease toward zero, preventing an epidemic from getting started. Therefore, 60% of the population must be vaccinated to prevent an epidemic from getting started.

3. **(a)** To guarantee that $dI/dt < 0$, we must have $\alpha SI - \beta I < 0$. Factoring, we obtain

$$(\alpha S - \beta)I < 0,$$

and since I is positive, we have $\alpha S - \beta < 0$. In other words,

$$S < \frac{\beta}{\alpha}.$$

Including initial conditions for which $S(0) = \beta/\alpha$ is debatable since $S(0) = \beta/\alpha$ implies that $I(t)$ is decreasing for $t \geq 0$.

(b) If $S(0) < \beta/\alpha$, then $dI/dt < 0$. In that case, any initial influx of infecteds will decrease toward zero, and the epidemic will die out. The fraction vaccinated must be at least $1 - \beta/\alpha$.

4. **(a)** We have

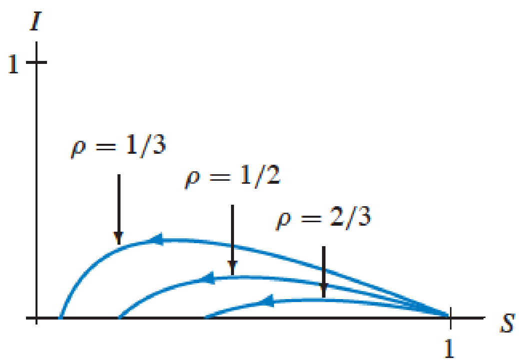
$$\frac{dI}{dS} = -1 + \frac{\rho}{S}.$$

Then $dI/dS = 0$ if and only if $S = \rho$. Furthermore, $d^2I/dt^2 = -\rho/S^2$ is always negative. By the Second Derivative Test, we conclude that the maximum value of $I(S)$ occurs at $S = \rho$. Evaluating $I(S)$ at $S = \rho$, we obtain the maximum value

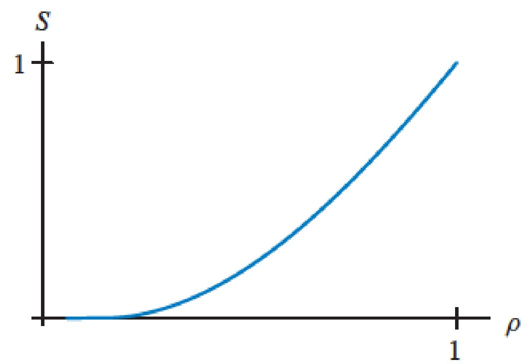
$$I(\rho) = 1 - \rho + \rho \ln \rho.$$

(b) For an epidemic to occur, $S(0) > \beta/\alpha$ (see Exercise 3). If $\beta > \alpha$, then $\beta/\alpha > 1$. Therefore, for an epidemic to occur under these conditions, $S(0) > 1$, which is not possible since $S(t)$ is defined as a proportion of the total population.

5. **(a)**



(b)



(c) As ρ increases, the limit of $S(t)$ as $t \to \infty$ approaches 1. Therefore, as ρ increases, the fraction of the population that contract the disease approaches zero.

6. **(a)** Note that

$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = (-\alpha SI + \gamma R) + (\alpha SI - \beta I) + (\beta I - \gamma R)$$
$$= 0$$

for all t.

(b) If we substitute $R = 1 - (S + I)$ into dS/dt, we get

$$\frac{dS}{dt} = -\alpha SI + \gamma(1 - (S + I))$$
$$\frac{dI}{dt} = \alpha SI - \beta I.$$

(c) If $dI/dt = 0$, then either $I = 0$ or $S = \beta/\alpha$.

If $I = 0$, then $dS/dt = \gamma(1 - S)$, which is zero if $S = 1$. We obtain the equilibrium point $(S, I) = (1, 0)$.

If $S = \beta/\alpha$, we set $dS/dt = 0$, and therefore,

$$-\alpha\left(\frac{\beta}{\alpha}\right)I + \gamma\left(1 - \left(\frac{\beta}{\alpha} + I\right)\right) = 0$$

$$-\beta I + \gamma - \frac{\gamma\beta}{\alpha} - \gamma I = 0$$

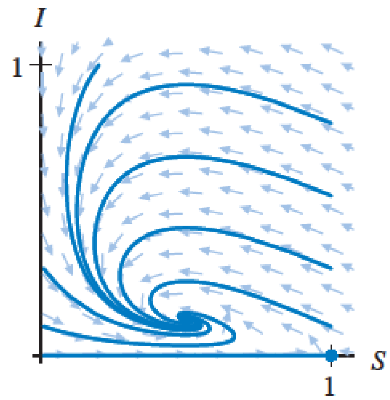
$$\frac{\gamma(\alpha - \beta)}{\alpha} = (\beta + \gamma)I,$$

so

$$I = \frac{\gamma(\alpha - \beta)}{\alpha(\beta + \gamma)}.$$

Therefore, there exists another equilibrium point $(S, I) = \left(\frac{\beta}{\alpha}, \frac{\gamma(\alpha - \beta)}{\alpha(\beta + \gamma)}\right)$.

(d)



Given $\alpha = 0.3$, $\beta = 0.15$, and $\gamma = 0.05$, the equilibrium points are $(S, I) = (1, 0)$ and $(S, I) = (0.5, 0.125)$ (see part (b)). For any solution with $I(0) = 0$, the solution tends toward $(1, 0)$, which corresponds to a population where no one ever becomes infected. For all other initial conditions, the solutions tend toward $(0.5, 1.25)$ as t approaches infinity.

(e) We fix $\alpha = 0.3$ and $\beta = 0.15$. If γ is slightly greater than 0.05, the equilibrium point

$$(S, I) = \left(0.5, \frac{0.15\gamma}{0.15 + \gamma}\right)$$

shifts vertically upward, corresponding to a larger proportion of the population being infected as $t \to \infty$. For γ slightly less than 0.05, the same equilibrium point shifts vertically downward, corresponding to a smaller proportion of the population being infected as $t \to \infty$.

7. **(a)** If $I = 0$, both equations are zero, so the S-axis consists entirely of equilibrium points. If $I \neq 0$, then S would have to be zero. However, in that case, the second equation reduces to $dI/dt = -\beta I$, which cannot be zero by assumption. Therefore, all equilibrium points must lie on the S-axis.

(b) We have $dI/dt > 0$ if and only if $\alpha S\sqrt{I} - \beta I > 0$. Factoring out $\sqrt{I}$, we obtain

$$(\alpha S - \beta\sqrt{I})\sqrt{I} > 0.$$

Since $\sqrt{I} \geq 0$, we have

$$\alpha S - \beta\sqrt{I} > 0$$

$$-\beta\sqrt{I} > \alpha S$$

$$\sqrt{I} < -\frac{\alpha}{\beta} S$$

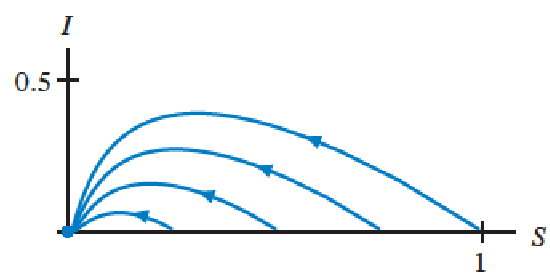
$$I < \left(\frac{\alpha}{\beta}\right)^2 S^2.$$

The resulting region is bounded by the S-axis and the parabola

$$I = \left(\frac{\alpha S}{\beta}\right)^2,$$

and lies in the half-plane $I > 0$.

(c) The model predicts that the entire population will become infected. That is, $R(t) \to 1$ as $t \to \infty$.



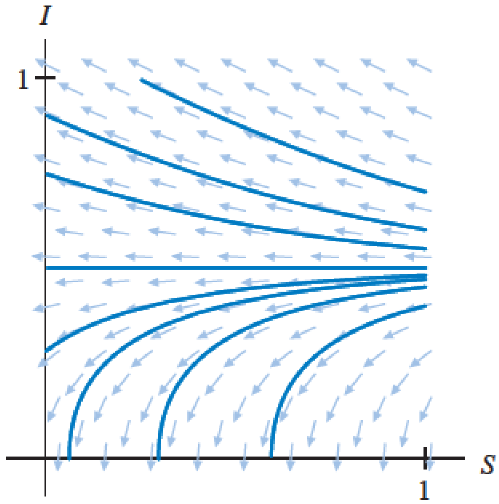
8. **(a)** Factoring the right-hand side of the equation for dI/dt, we get

$$\frac{dI}{dt} = (\alpha I - \gamma)S.$$

Therefore, the line $S = 0$ (the I-axis) is a line of equilibrium points. If $S \neq 0$, then $dI/dt = 0$ only if $I = \gamma/\alpha$. However, if $S \neq 0$ and $I = \gamma/\alpha$, then $dS/dt \neq 0$. So there are no other equilibrium points.

(b) If $S \neq 0$, then S is positive. Therefore, $dI/dt > 0$ if and only if $\alpha I - \gamma > 0$ and $S > 0$. In other words $dI/dt > 0$ if and only if $I > \gamma/\alpha$ and $S > 0$.

(c)



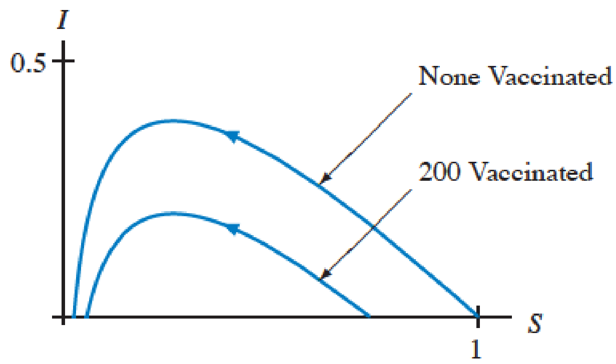
The model predicts that if $I(0) > 0.5$, then the infected (zombie) population will grow until there are no more susceptibles. If $I(0) = 0.5$, then the infected population will remain constant for all time. If $I(0) < 0.5$, then the entire infected population will die out over time.

9. **(a)** $\beta = 0.44$.

(b) As $t \to \infty$, $S(t) \approx 19$. Therefore, the total number of infected students is 744.

(c) Since β determines how quickly students move from being infected to recovered, a small value of β relative to α indicates that it will take a long time for the infected students to recover.

10. With 200 students vaccinated, there are only 563 students who can potentially contract the disease. The total population of students is still 763 students, but the vaccinated students decrease the interaction between infecteds and susceptibles. Starting with one infected student, we have $(S(0), I(0)) \approx (0.737, 0.001)$.



Note that if 200 students are vaccinated, the maximum of $I(t)$ is smaller. Consequently, the maximum number of infecteds is smaller if 200 students are vaccinated. More specifically, if none of the students are vaccinated, the maximum of $I(t)$ is approximately 293 students. If 200 students are vaccinated, the maximum of $I(t)$ is approximately 155 students.

In addition, the total number of students who catch the disease decreases if 200 students are initially vaccinated. More specifically, if none of the students are vaccinated, $S(t)$ is approximately 19 as $t \to \infty$. Thus, the total number of students infected is $763 - 19 = 744$ students. If 200 students are initially vaccinated, $S(t) \approx 42$ as $t \to \infty$. Thus, the total number of students infected is $563 - 42 = 521$ students.