1. (16 points) Consider the second-order equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x + x^3 = 0.$$

Convert this differential equation to a first-order system and calculate its equilibrium point(s).

Let $\frac{dx}{dt} = v$.

Then $\frac{dv}{dt} = \frac{d^2x}{dt^2} = -2\frac{dx}{dt} + 3x - x^3$

$= -2v + 3x - x^3$.

First-order system:

$\frac{dx}{dt} = v$

$\frac{dv}{dt} = -2v + 3x - x^3$

Equilibrium points: $\frac{dx}{dt} = \frac{dv}{dt} = 0$

$\Rightarrow v = 0 \Rightarrow 3x - x^3 = 0$

$x(3 - x^2) = 0$

Three equilibrium points:

$(x, v) = (0, 0)$ $(x, v) = (\sqrt{3}, 0)$

$(x, v) = (-\sqrt{3}, 0)$

2. (20 points) Consider the linear system

$$\frac{d\mathbf{Y}}{dt} = \underbrace{\begin{pmatrix} 4 & 0 \\ 1 & 4 \end{pmatrix}}_{= A} \mathbf{Y}.$$

(a) Find *all* straight-line solutions. Make sure that you show the computations that justify your answers.

$\det(A - \lambda I) = (4-\lambda)^2$ $\lambda = 4$ repeated evalue

$\lambda = 4$ eigenvecs:

$\begin{cases} 4x_0 = 4x_0 \\ x_0 + 4y_0 = 4y_0 \end{cases} \Rightarrow x_0 = 0 \Leftrightarrow$ y-axis.

$\mathbf{Y}(t) = k e^{4t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ k is any nonzero constant

(b) Solve the initial-value problem where $\mathbf{Y}(0) = (3, 2)$.

Partially decoupled but we use the guessing method. Guess

$\mathbf{Y}(t) = e^{4t}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + te^{4t}\mathbf{V}_1$.

Then $\mathbf{V}_1 = (A - 4I)\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$\Rightarrow \mathbf{Y}(t) = e^{4t}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + te^{4t}\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

3. (21 points) Are the following statements true or false? **You will not receive any credit unless you justify your answers.**

(a) The function $(x(t), y(t)) = (2e^{-3t}, e^{-6t})$ is a solution to the system of differential equations $dx/dt = -3x$ and $dy/dt = -2x^2 - 2y$.

False

No wrong answers:

$x(t) = 2e^{-3t} \Rightarrow \frac{dx}{dt} = -6e^{-3t} = -3x(t)$ ✓

$y(t) = e^{-6t} \Rightarrow \frac{dy}{dt} = -6e^{-6t}$ ← do not agree

$-2x^2 - 2y = -2(2e^{-3t})^2 - 2(e^{-6t}) = -10e^{-6t}$

(b) If $\mathbf{Y}_0$ is an eigenvector for a matrix $\mathbf{A}$, then so is any nonzero scalar multiple of $\mathbf{Y}_0$.

True. Let $AY_0 = \lambda Y_0$ and let r be any scalar. Then

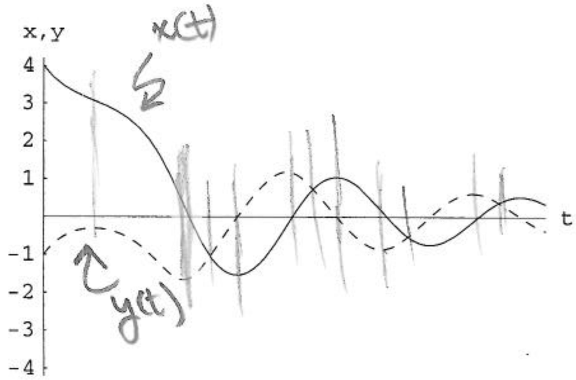
$A(rY_0) = r(AY_0) = r(\lambda Y_0)$
$= \lambda(rY_0)$ ✓

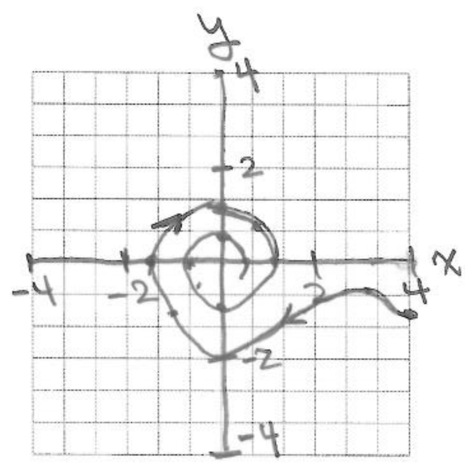
(c) If the function $(x_1(t), y_1(t)) = (\cos t, \sin t)$ is a solution to an autonomous first-order system, then the function $(x_2(t), y_2(t)) = (\cos(t-2), \sin(t-2))$ is also a solution to the same system.

True The second function sweeps out the same solution curve as the first function two units of time behind the first. This is how solutions to autonomous systems behave if their curves have one point in common.

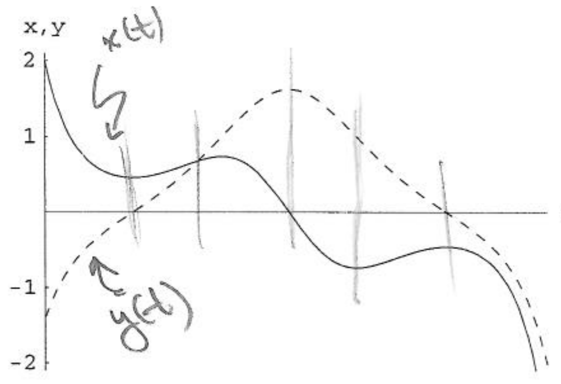
4. (20 points) In each part of this problem, $x(t)$- and $y(t)$-graphs for a solution are shown. The $x(t)$-graph is the solid curve, and the $y(t)$-graph is the dashed curve. Using the graph paper on the right, sketch the solution curve corresponding to these graphs and indicate the direction the solution goes at t increases by placing at least one arrowhead on your curve. Make sure that the axes in your drawing are clearly labeled with a variable and a scale. Sketch only that part of the solution curve that corresponds to the interval of t-values shown in the graphs.

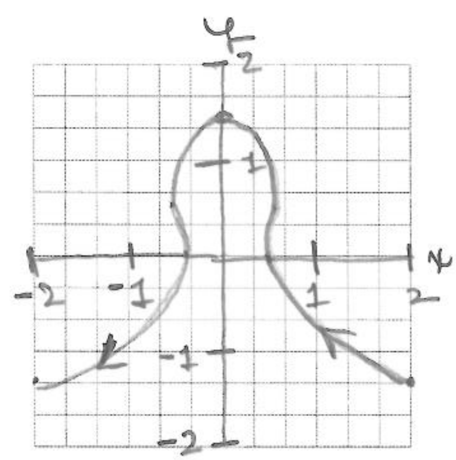
(a)





(b)





5. (20 points) Consider the one-parameter family of linear systems

$$\frac{dx}{dt} = ax - y$$
$$\frac{dy}{dt} = 2x$$

where a is the parameter. For what values of a is the equilibrium point at the origin a spiral sink?

$A = \begin{pmatrix} a & -1 \\ 2 & 0 \end{pmatrix}$ $\quad \det(A - \lambda I) = \det\begin{pmatrix} a-\lambda & -1 \\ 2 & -\lambda \end{pmatrix}$

$= \lambda(\lambda - a) + 2$

$= \lambda^2 - a\lambda + 2$

The eigenvalues are

$$\frac{a \pm \sqrt{a^2 - 8}}{2}.$$

The origin is a spiral sink $\Leftrightarrow$ the evals are complex ($a^2 - 8 < 0$) and have negative real part ($a < 0$).

$a^2 - 8 < 0 \Leftrightarrow a^2 < 8$

$\Leftrightarrow -2\sqrt{2} < a < 2\sqrt{2}$.

spiral sink $\Leftrightarrow -2\sqrt{2} < a < 0$.