

2. (17 points) Find the solution (in scalar form) of the initial-value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 5y = 0, \quad y(0) = 11, \quad y'(0) = -7.$$

$$\text{Char eqn: } \lambda^2 + 4\lambda - 5 = 0$$

$$(\lambda + 5)(\lambda - 1) = 0$$

Gen soln:

$$y(t) = k_1 e^{-5t} + k_2 e^t$$

Initial-value problem:

$$y(0) = 11 = k_1 + k_2$$

$$y'(t) = -5k_1 e^{-5t} + k_2 e^t$$

$$y'(0) = -7 = -5k_1 + k_2$$

$$\Rightarrow 18 = 6k_1 \Rightarrow k_1 = 3$$

$$\Rightarrow k_2 = 8$$

$$y(t) = 3e^{-5t} + 8e^t.$$

3. (20 points) Consider the linear system

$$\begin{aligned}\frac{dx}{dt} &= -4x - y \\ \frac{dy}{dt} &= 4x.\end{aligned}$$

$$A = \begin{pmatrix} -4 & -1 \\ 4 & 0 \end{pmatrix}$$

- (a) Sketch the phase portrait of this system over the square $-6 \leq x \leq 6$ and $-6 \leq y \leq 6$. Make sure that you show the computations that justify your answers.

$$\det(A - \lambda I) = \det \begin{pmatrix} -4 - \lambda & -1 \\ 4 & -\lambda \end{pmatrix}$$

$$= \lambda(4 + \lambda) + 4$$

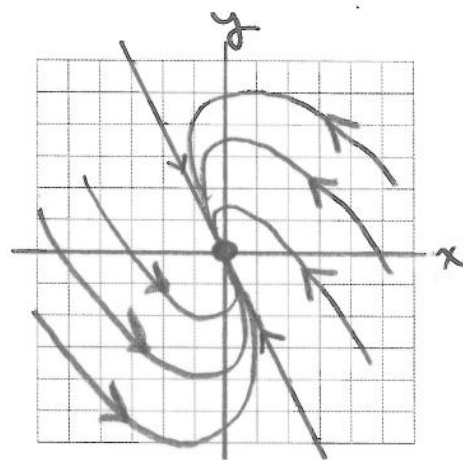
$$= \lambda^2 + 4\lambda + 4$$

$$= (\lambda + 2)^2$$

$$\lambda = -2 \text{ repeated}$$

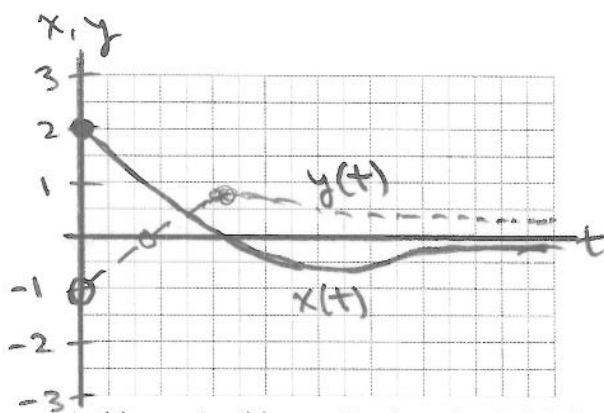
evecs

$$\begin{cases} -4x - y = -2x \\ 4x = -2y \end{cases} \Rightarrow y = -2x$$

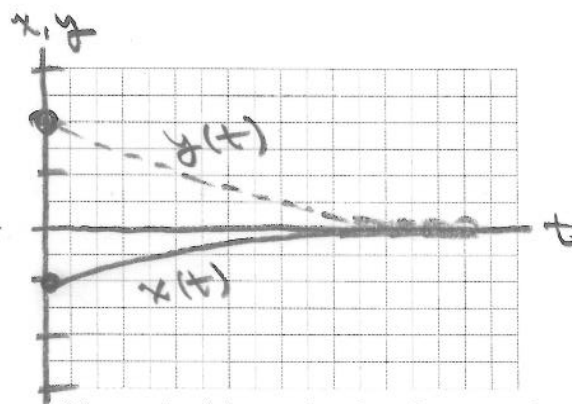


vec @ $(1, 0)$ is $(-4, 4)$

- (b) Plot the $x(t)$ - and $y(t)$ -graphs for $t \geq 0$ for the initial conditions given below. Make sure that you label the axes, indicate a scale on the vertical axis, and distinguish the $x(t)$ - from the $y(t)$ -graph.



$x(t)$ - and $y(t)$ -graphs for the initial condition $(2, -1)$



$x(t)$ - and $y(t)$ -graphs for the initial condition $(-1, 2)$

↑ evec \Rightarrow
SL soln

4. (20 points) Consider the initial-value problem

$$\frac{d^2 y}{dt^2} + \left| \frac{dy}{dt} \right| \frac{dy}{dt} + 3y = 0, \quad (y_0, v_0) = (1, 0).$$

- (a) Reduce this second-order equation to a first-order system and use Euler's method with 4 steps to calculate an approximate solution over the interval $0 \leq t \leq 2$. Enter the results in the table provided. Make sure that you show enough calculations so that the grader can understand how you obtained your answer. You may use a calculator and do all calculations to two decimal places if you wish.

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = \frac{d^2 y}{dt^2} = -3y - |v|v$$

$$\text{Euler: } \Delta t = \frac{2}{4} = .5$$

$$y_{k+1} = y_k + (\Delta t)(v_k)$$

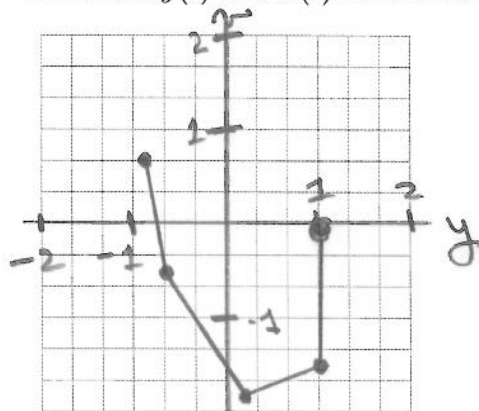
$$v_{k+1} = v_k + (\Delta t)(-3y_k - |v_k|v_k)$$

k	y_k	v_k
0	1	0
1	1	-1.5
2	.25	-1.88
3	-.69	-.49
4	-.94	.67

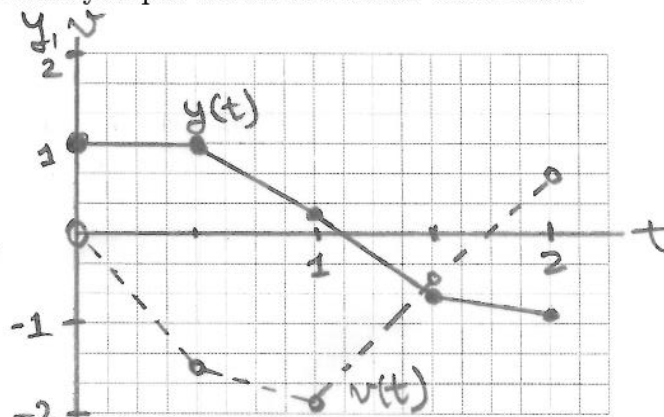
$$y_1 = y_0 + (.5)(0) = 1$$

$$v_1 = v_0 + (.5)(-3) = -1.5$$

- (b) On the left-hand grid, sketch the approximate solution curve in the yv -phase plane, and on the right-hand grid, sketch the graphs of the approximations to the functions $y(t)$ and $v(t)$. Make sure that you put labels and scales on all axes.



solution curve in yv -plane



$y(t)$ - and $v(t)$ -graphs

eigenvalue calculations on next page.

5. (20 points) On the next page, there are four matrices that can be used to form linear systems. There are also $x(t)$ - and $y(t)$ -graphs of two solutions to those systems. Pair each solution with its corresponding matrix, and provide a brief justification for your choice. **You will not receive any credit unless you provide a valid justification.**

(i) The matrix for solution 1 is B. My reason for choosing this answer is:

Nonzero equilibrium \Rightarrow B or D

$$x + 2y = 0 \Rightarrow B$$

(ii) The matrix for solution 2 is A. My reason for choosing this answer is:

Purely imaginary eigenvalues \Rightarrow A or C

$$\text{Period} = \frac{2\pi}{3} \Rightarrow \lambda = \pm 3i \\ \Rightarrow A$$

5. (continued) Answer this question on the previous page.

The four matrices:

$$A = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 1-\lambda & 2 \\ -5 & -1-\lambda \end{pmatrix} \\ &= (\lambda-1)(\lambda+1) + 10 \\ &= \lambda^2 + 9 \Rightarrow \lambda = \pm 3i \end{aligned}$$

$$B = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$$

$$\begin{aligned} \det(B - \lambda I) &= (1-\lambda)(-2-\lambda) + 2 \\ &= (\lambda-1)(\lambda+2) + 2 \\ &= \lambda^2 + \lambda \Rightarrow \\ &\lambda = 0, -1 \end{aligned}$$

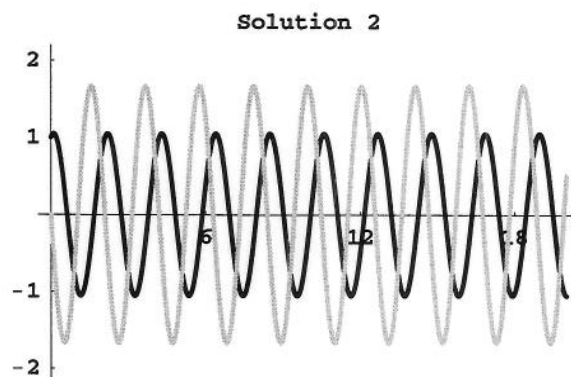
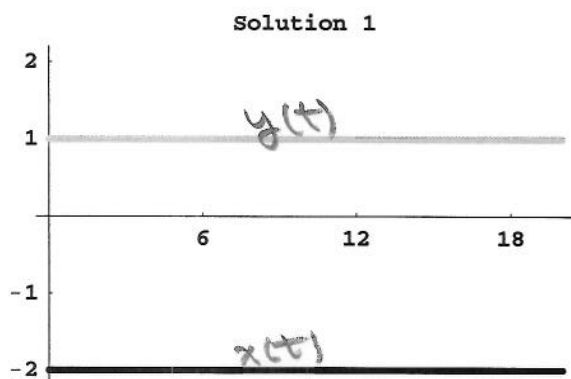
$$C = \begin{pmatrix} 1 & -1 \\ 5 & -1 \end{pmatrix}$$

$$\begin{aligned} \det(C - \lambda I) &= (1-\lambda)(-1-\lambda) + 5 \\ &= (\lambda-1)(\lambda+1) + 5 \\ &= \lambda^2 + 4 \\ &\Rightarrow \lambda = \pm 2i \end{aligned}$$

$$D = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}$$

$$\begin{aligned} \det(D - \lambda I) &= (2-\lambda)(-1-\lambda) + 2 \\ &= (\lambda-2)(\lambda+1) + 2 \\ &= \lambda^2 - \lambda \\ &\lambda = 0, 1 \end{aligned}$$

Here are $x(t)$ - and $y(t)$ -graphs for the two solutions. The $x(t)$ -graph is the solid black curve, and the $y(t)$ -graph is the shaded curve.



equilibrium
solution

$$(x(t), y(t)) = (-2, 1)$$