

1. (20 points) Consider the linear system

$$\frac{dY}{dt} = \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix} Y.$$

$\equiv A$

- (a) Find *all* straight-line solutions. Make sure that you show the computations that justify your answers.

$$\det(A - \lambda I) = (3 - \lambda)^2 \quad \lambda = 3 \text{ repeated eigenvalue}$$

$\lambda = 3$ eigenvectors:

$$\begin{cases} 3x_0 = 3x_0 \\ x_0 + 3y_0 = 3y_0 \end{cases} \Rightarrow x_0 = 0 \Leftrightarrow y\text{-axis}$$

$$Y(t) = k e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad k \text{ is any nonzero constant.}$$

- (b) Solve the initial-value problem where $Y(0) = (5, 3)$.

Partially decoupled but we use the guessing method. Guess

$$Y(t) = e^{3t} \begin{pmatrix} 5 \\ 3 \end{pmatrix} + t e^{3t} V_1$$

$$\text{Then } V_1 = (A - 3I) \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

$$\Rightarrow Y(t) = e^{3t} \begin{pmatrix} 5 \\ 3 \end{pmatrix} + t e^{3t} \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

2. (16 points) Consider the second-order equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - 5x + x^3 = 0.$$

Convert this differential equation to a first-order system and calculate its equilibrium point(s).

$$\text{Let } \frac{dx}{dt} = v.$$

$$\begin{aligned}\text{Then } \frac{dv}{dt} &= \frac{d^2x}{dt^2} = -3\frac{dx}{dt} + 5x - x^3 \\ &= -3v + 5x - x^3.\end{aligned}$$

First-order system:

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = -3v + 5x - x^3.$$

Equilibrium points: $\frac{dx}{dt} = \frac{dv}{dt} = 0$

$$\Rightarrow v = 0 \Rightarrow 5x - x^3 = 0$$

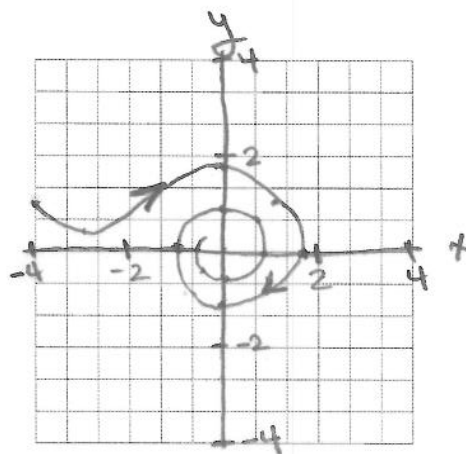
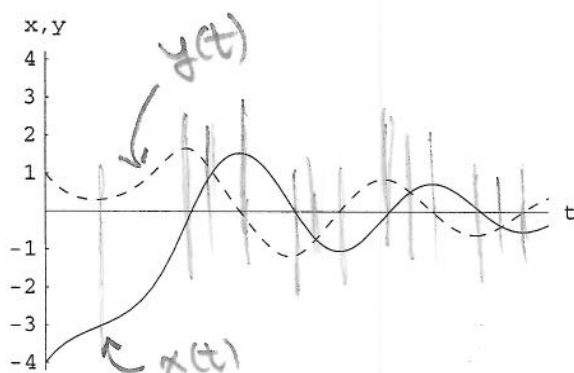
$$x(5 - x^2) = 0$$

Three equilibrium points.

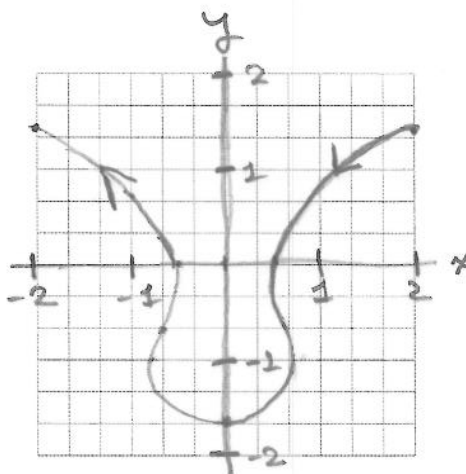
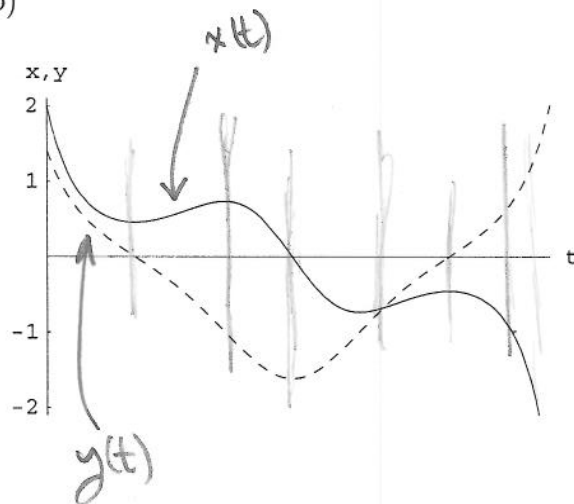
$$(x, v) = (0, 0) \quad (x, v) = (-\sqrt{5}, 0) \quad (x, v) = (\sqrt{5}, 0)$$

3. (20 points) In each part of this problem, $x(t)$ - and $y(t)$ -graphs for a solution are shown. The $x(t)$ -graph is the solid curve, and the $y(t)$ -graph is the dashed curve. Using the graph paper on the right, sketch the solution curve corresponding to these graphs and indicate the direction the solution goes as t increases by placing at least one arrowhead on your curve. Make sure that the axes in your drawing are clearly labeled with a variable and a scale. Sketch only that part of the solution curve that corresponds to the interval of t -values shown in the graphs.

(a)



(b)



4. (21 points) Are the following statements true or false? **You will not receive any credit unless you justify your answers.**

- (a) If \mathbf{Y}_0 is an eigenvector for a matrix \mathbf{A} , then so is any nonzero scalar multiple of \mathbf{Y}_0 .

True. Let $\mathbf{A}\mathbf{Y}_0 = \lambda\mathbf{Y}_0$ and let r be any scalar. Then

$$\mathbf{A}(r\mathbf{Y}_0) = r(\mathbf{A}\mathbf{Y}_0) = r(\lambda\mathbf{Y}_0) = \lambda(r\mathbf{Y}_0) \quad \checkmark$$

- (b) The function $(x(t), y(t)) = (3e^{-2t}, e^{-4t})$ is a solution to the system of differential equations $dx/dt = -2x$ and $dy/dt = -x^2 + 5y$.

True

No wrong answers:

$$x(t) = 3e^{-2t} \rightarrow \frac{dx}{dt} = -6e^{-2t} = -2(3e^{-2t}) = -2x(t) \quad \checkmark$$

$$y(t) = e^{-4t} \Rightarrow \frac{dy}{dt} = -4e^{-4t}$$

$$\begin{aligned} -x^2 + 5y &= -(3e^{-2t})^2 + 5e^{-4t} \\ &= -9e^{-4t} + 5e^{-4t} = -4e^{-4t} \quad \checkmark \end{aligned}$$

- (c) If the function $(x_1(t), y_1(t)) = (\cos t, \sin t)$ is a solution to an autonomous first-order system, then the function $(x_2(t), y_2(t)) = (\cos(t-3), \sin(t-3))$ is also a solution to the same system.

True. The second function sweeps out the same curve in the phase plane as the first function three units of time behind the first. This is how solutions to autonomous systems behave if their curves intersect.

5. (20 points) Consider the one-parameter family of linear systems

$$\begin{aligned}\frac{dx}{dt} &= ax + 3y \\ \frac{dy}{dt} &= -x\end{aligned}$$

where a is the parameter. For what values of a is the equilibrium point at the origin a spiral sink?

$$\begin{aligned}A &= \begin{pmatrix} a & 3 \\ -1 & 0 \end{pmatrix} & \det(A - \lambda I) &= \det \begin{pmatrix} a - \lambda & 3 \\ -1 & -\lambda \end{pmatrix} \\ & & &= \lambda(\lambda - a) + 3 \\ & & &= \lambda^2 - a\lambda + 3\end{aligned}$$

The eigenvalues are

$$\frac{a \pm \sqrt{a^2 - 12}}{2}$$

The origin is a spiral sink \Leftrightarrow
the evals are complex ($a^2 - 12 < 0$)
and have negative real part ($a < 0$).

$$a^2 - 12 < 0 \Leftrightarrow a^2 < 12$$

$$\Leftrightarrow -\sqrt{12} < a < \sqrt{12}$$

$$\text{spiral sink} \Leftrightarrow -\sqrt{12} < a < 0.$$