1. Using Euler's formula, we can write the complex-valued solution $\mathbf{Y}_c(t)$ as

$$\begin{aligned}\mathbf{Y}_c(t) &= e^{(1+3i)t}\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\ &= e^t e^{3it}\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\ &= e^t(\cos 3t + i\sin 3t)\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\ &= e^t\begin{pmatrix} 2\cos 3t - \sin 3t \\ \cos 3t \end{pmatrix} + ie^t\begin{pmatrix} 2\sin 3t + \cos 3t \\ \sin 3t \end{pmatrix}.\end{aligned}$$

Hence, we have

$$\mathbf{Y}_{\mathrm{re}}(t) = e^t\begin{pmatrix} 2\cos 3t - \sin 3t \\ \cos 3t \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_{\mathrm{im}}(t) = e^t\begin{pmatrix} \cos 3t + 2\sin 3t \\ \sin 3t \end{pmatrix}.$$

The general solution is

$$\mathbf{Y}(t) = k_1 e^t\begin{pmatrix} 2\cos 3t - \sin 3t \\ \cos 3t \end{pmatrix} + k_2 e^t\begin{pmatrix} \cos 3t + 2\sin 3t \\ \sin 3t \end{pmatrix}.$$

2. The complex solution is

$$\mathbf{Y}_c(t) = e^{(-2+5i)t}\begin{pmatrix} 1 \\ 4-3i \end{pmatrix},$$

so we can use Euler's formula to write

$$\begin{aligned}
\mathbf{Y}_c(t) &= e^{(-2+5i)t}\begin{pmatrix} 1 \\ 4-3i \end{pmatrix} \\
&= e^{-2t}e^{5it}\begin{pmatrix} 1 \\ 4-3i \end{pmatrix} \\
&= e^{-2t}(\cos 5t + i\sin 5t)\begin{pmatrix} 1 \\ 4-3i \end{pmatrix} \\
&= e^{-2t}\begin{pmatrix} \cos 5t \\ 4\cos 5t + 3\sin 5t \end{pmatrix} + ie^{-2t}\begin{pmatrix} \sin 5t \\ 4\sin 5t - 3\cos 5t \end{pmatrix}.
\end{aligned}$$

Hence, we have

$$\mathbf{Y}_{\text{re}}(t) = e^{-2t}\begin{pmatrix} \cos 5t \\ 4\cos 5t + 3\sin 5t \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_{\text{im}}(t) = e^{-2t}\begin{pmatrix} \sin 5t \\ 4\sin 5t - 3\cos 5t \end{pmatrix}.$$

The general solution is

$$\mathbf{Y}(t) = k_1e^{-2t}\begin{pmatrix} \cos 5t \\ 4\cos 5t + 3\sin 5t \end{pmatrix} + k_2e^{-2t}\begin{pmatrix} \sin 5t \\ 4\sin 5t - 3\cos 5t \end{pmatrix}.$$

3. **(a)** The characteristic equation is

$$(-\lambda)^2 + 4 = \lambda^2 + 4 = 0,$$

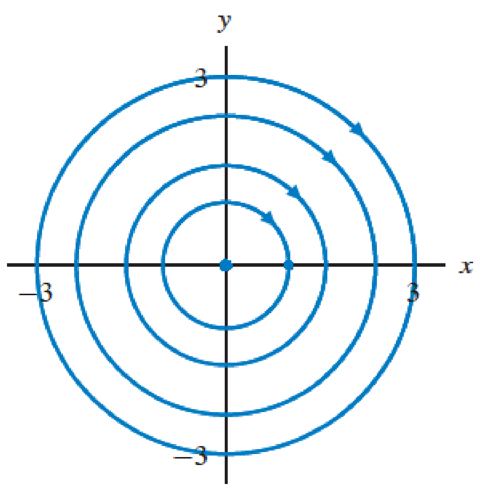
and the eigenvalues are $\lambda = \pm 2i$.

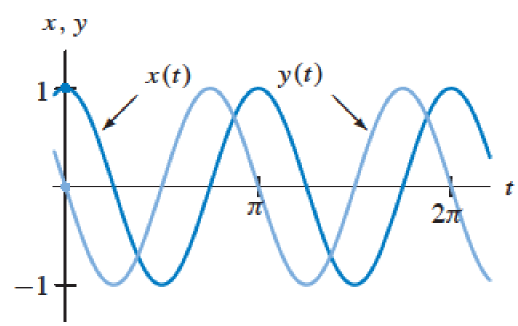
(b) Since the real part of the eigenvalues are 0, the origin is a center.

(c) Since $\lambda = \pm 2i$, the natural period is $2\pi/2 = \pi$, and the natural frequency is $1/\pi$.

(d) At $(1, 0)$, the tangent vector is $(-2, 0)$. Therefore, the direction of oscillation is clockwise.

(e) According to the phase plane, $x(t)$ and $y(t)$ are periodic with period π. At the initial condition $(1, 0)$, both $x(t)$ and $y(t)$ are initially decreasing.





4. **(a)** The characteristic equation is

$$(2-\lambda)(6-\lambda)+8=\lambda^2-8\lambda+20,$$

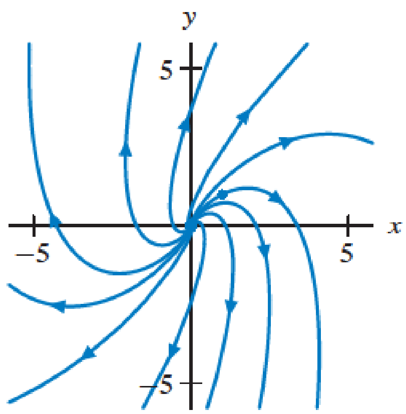
and the eigenvalues are $\lambda = 4 \pm 2i$.

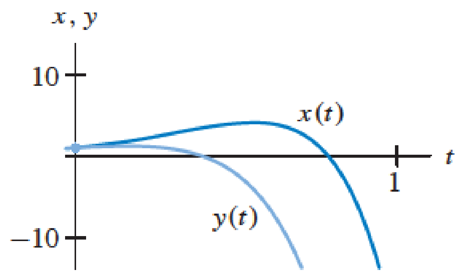
(b) Since the real part of the eigenvalues is positive, the origin is a spiral source.

(c) Since $\lambda = 4 \pm 2i$, the natural period is $2\pi/2 = \pi$, and the natural frequency is $1/\pi$.

(d) At the point $(1, 0)$, the tangent vector is $(2, -4)$. Therefore, the solution curves spiral around the origin in a clockwise fashion.

(e) Since $d\mathbf{Y}/dt = (4, 2)$ at $\mathbf{Y}_0 = (1, 1)$, both $x(t)$ and $y(t)$ increase initially. The distance between successive zeros is π, and the amplitudes of both $x(t)$ and $y(t)$ are increasing.





5. **(a)** The characteristic polynomial is

$$(-3-\lambda)(1-\lambda)+15=\lambda^2+2\lambda+12,$$

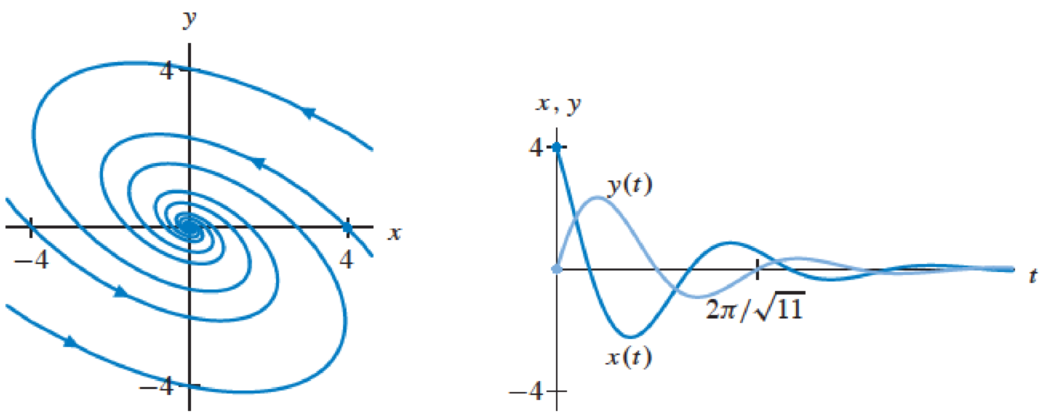
so the eigenvalues are $\lambda=-1\pm i\sqrt{11}$.

(b) The eigenvalues are complex and the real part is negative, so the origin is a spiral sink.

(c) The natural period is $2\pi/\sqrt{11}$. The natural frequency is $\sqrt{11}/(2\pi)$.

(d) At the point $(1,0)$, the vector field is $(-3,3)$. Hence, the solution curves must spiral in a counterclockwise fashion.

(e)



6. **(a)** The characteristic polynomial is

$$(-\lambda)(-1-\lambda)+4=\lambda^2+\lambda+4,$$

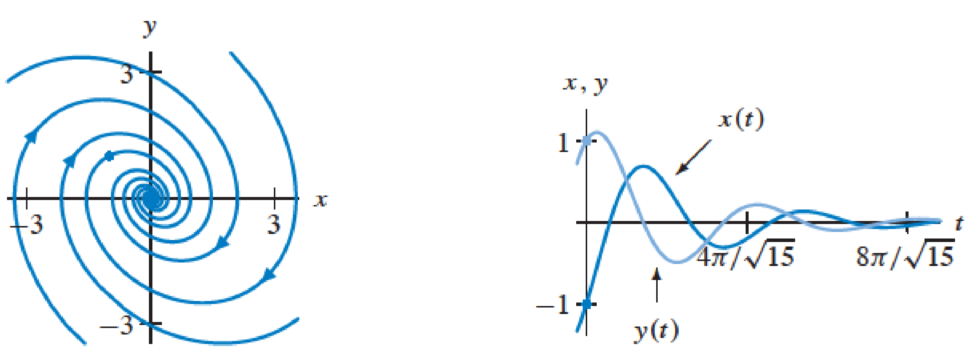
so the eigenvalues are $\lambda=(-1\pm i\sqrt{15})/2$.

(b) The eigenvalues are complex and the real part is negative, so the origin is a spiral sink.

(c) The natural period is $2\pi/(\sqrt{15}/2)=4\pi/\sqrt{15}$. The natural frequency is $\sqrt{15}/(4\pi)$.

(d) The vector field at $(1,0)$ is $(0,-2)$. Hence, solution curves spiral about the origin in a clockwise fashion.

(e) From the phase plane, we see that both $x(t)$ and $y(t)$ are initially increasing. However, $y(t)$ quickly reaches a local maximum. Although both functions oscillate, each successive oscillation has a smaller amplitude.



7. **(a)** The characteristic equation is

$$(2-\lambda)(1-\lambda)+12=\lambda^2+3\lambda+14=0,$$

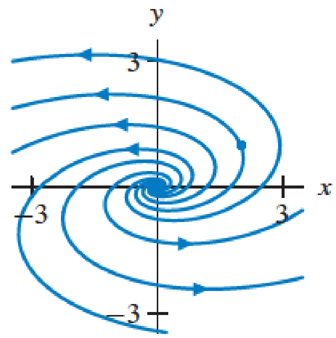
and the eigenvalues are $\lambda=(3\pm\sqrt{47}\,i)/2$.

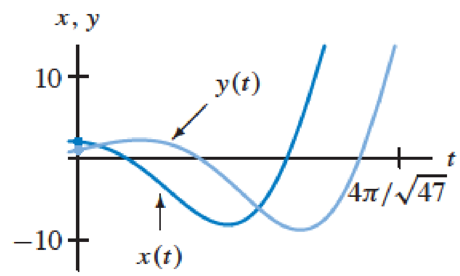
(b) Since the real part of the eigenvalues is positive, the origin is a spiral source.

(c) Since $\lambda=(3\pm\sqrt{47}i)/2$, natural period is $4\pi/\sqrt{47}$, and natural frequency is $\sqrt{47}/(4\pi)$.

(d) At the point $(1,0)$, the tangent vector is $(2,2)$. Therefore, the solution curves spiral about the origin in a counterclockwise fashion.

(e) From the phase plane, we see that both $x(t)$ and $y(t)$ oscillate and that the amplitude of these oscillations increases rapidly.





8. **(a)** The characteristic polynomial is

$$(1-\lambda)(2-\lambda)+12=\lambda^2-3\lambda+14,$$

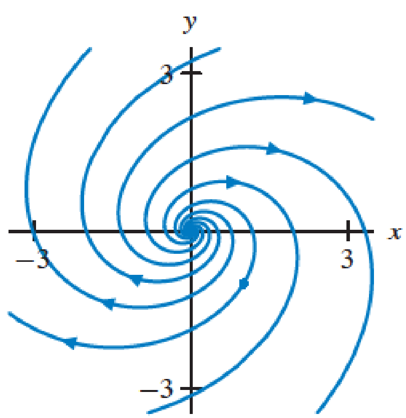
so the eigenvalues are $\lambda=(3\pm i\sqrt{47})/2$.

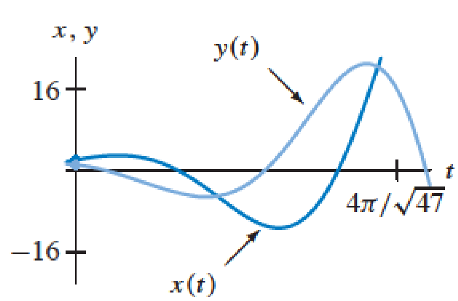
(b) The eigenvalues are complex and the real part is positive, so the origin is a spiral source.

(c) The natural period is $2\pi/(\sqrt{47}/2)=4\pi/\sqrt{47}$. The natural frequency is $\sqrt{47}/(4\pi)$.

(d) The vector field at $(1,0)$ is $(1,-3)$. Hence, the solution curves spiral about the origin in a clockwise fashion.

(e) From the phase plane, we see that both $x(t)$ and $y(t)$ oscillate about 0 and that the amplitude of these oscillations grows quickly.





9. **(a)** According to Exercise 3, $\lambda = \pm 2i$. The eigenvectors (x, y) associated to eigenvalue $\lambda = 2i$ must satisfy the equation $2y = 2ix$, which is equivalent to $y = ix$. One such eigenvector is $(1, i)$, and thus we have the complex solution

$$\mathbf{Y}(t) = e^{2it}\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} + i\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}.$$

Taking real and imaginary parts, we obtain the general solution

$$\mathbf{Y}(t) = k_1\begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} + k_2\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}.$$

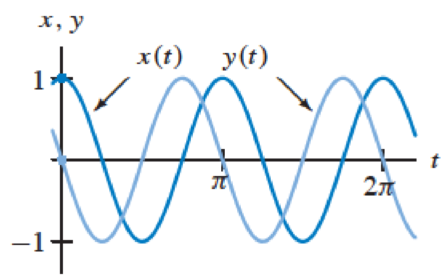
(b) From the initial condition, we obtain

$$k_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and therefore, $k_1 = 1$ and $k_2 = 0$. Consequently,the solution with the initial condition $(1, 0)$ is

$$\mathbf{Y}(t) = \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix}.$$

(c)



10. **(a)** According to Exercise 4, the eigenvalues are $\lambda = 4 \pm 2i$. The eigenvectors (x, y) associated to the eigenvalue $4 + 2i$ must satisfy the equation $y = (1 + i)x$. Hence, using the eigenvector $(1, 1 + i)$, we obtain the complex-valued solution

$$\mathbf{Y}(t) = e^{(4+2i)t} \begin{pmatrix} 1 \\ 1+i \end{pmatrix} = e^{4t} \begin{pmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{pmatrix} + i e^{4t} \begin{pmatrix} \sin 2t \\ \cos 2t + \sin 2t \end{pmatrix}.$$

From the real and imaginary parts of $\mathbf{Y}(t)$, we obtain the general solution

$$\mathbf{Y}(t) = k_1 e^{4t} \begin{pmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{pmatrix} + k_2 e^{4t} \begin{pmatrix} \sin 2t \\ \cos 2t + \sin 2t \end{pmatrix}.$$

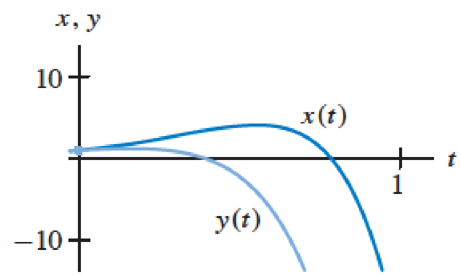
(b) Using the initial condition, we have

$$k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and thus $k_1 = 1$ and $k_2 = 0$. The desired solution is

$$\mathbf{Y}(t) = e^{4t} \begin{pmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{pmatrix}.$$

(c)



11. **(a)** To find the general solution, we find the eigenvectors from the characteristic polynomial

$$(-3-\lambda)(1-\lambda)+15=\lambda^2+2\lambda+12.$$

The eigenvalues are $\lambda = -1 \pm i\sqrt{11}$. To find an eigenvector associated to the eigenvector $-1+i\sqrt{11}$, we must solve the equations

$$\begin{cases} -3x-5y = \left(-1+i\sqrt{11}\right)x \\ 3x+y = \left(-1+i\sqrt{11}\right)y. \end{cases}$$

We see that the eigenvectors must satisfy the equation $3x = (-2+i\sqrt{11})y$. Using the eigenvector $(-2+i\sqrt{11}, 3)$, we obtain the complex-valued solution

$$\mathbf{Y}(t) = e^{\left(-1+i\sqrt{11}\right)t}\begin{pmatrix} -2+i\sqrt{11} \\ 3 \end{pmatrix}.$$

Using Euler's formula, we write $\mathbf{Y}(t)$ as

$$\mathbf{Y}(t) = e^{-t}\left(\cos\sqrt{11}\,t + i\sin\sqrt{11}\,t\right)\begin{pmatrix} -2+i\sqrt{11} \\ 3 \end{pmatrix},$$

which can be expressed as

$$\mathbf{Y}(t) = e^{-t}\begin{pmatrix} -2\cos\sqrt{11}\,t - \sqrt{11}\sin\sqrt{11}\,t \\ 3\cos\sqrt{11}\,t \end{pmatrix} + ie^{-t}\begin{pmatrix} \sqrt{11}\cos\sqrt{11}\,t - 2\sin\sqrt{11}\,t \\ 3\sin\sqrt{11}\,t \end{pmatrix}.$$

Taking real and imaginary parts, we can form the general solution

$$k_1e^{-t}\begin{pmatrix} -2\cos\sqrt{11}\,t - \sqrt{11}\sin\sqrt{11}\,t \\ 3\cos\sqrt{11}\,t \end{pmatrix} + k_2e^{-t}\begin{pmatrix} \sqrt{11}\cos\sqrt{11}\,t - 2\sin\sqrt{11}\,t \\ 3\sin\sqrt{11}\,t \end{pmatrix}.$$

(b) To find the particular solution with initial condition $(4, 0)$, we solve for k_1 and k_2 and obtain

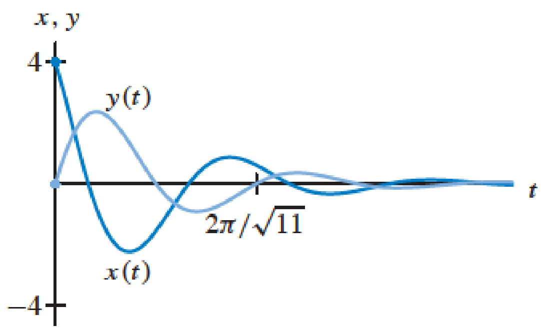
$$\begin{cases} -2k_1 + \sqrt{11}\,k_2 = 4 \\ 3k_1 = 0. \end{cases}$$

We have $k_1 = 0$ and $k_2 = 4/\sqrt{11}$.

The desired solution is

$$\mathbf{Y}(t) = e^{-t}\begin{pmatrix} 4\cos\sqrt{11}\,t - \frac{8}{\sqrt{11}}\sin\sqrt{11}\,t \\ \frac{12}{\sqrt{11}}\sin\sqrt{11}\,t \end{pmatrix}.$$

(c)



12. **(a)** The eigenvalues are the roots of the characteristic polynomial

$$(-\lambda)(-1-\lambda)+4=\lambda^2+\lambda+4.$$

So $\lambda = (-1 \pm i\sqrt{15})/2$. The eigenvectors (x, y) associated to the eigenvalue $\lambda = (-1+i\sqrt{15})/2$ must satisfy the equation $4y = (-1+i\sqrt{15})x$. Hence, $(4, -1+i\sqrt{15})$ is an eigenvector for this eigenvalue, and we have the complex-valued solution

$$\begin{aligned}
\mathbf{Y}(t) &= e^{(-1+i\sqrt{15})t/2}\begin{pmatrix} 4 \\ -1+i\sqrt{15} \end{pmatrix} \\
&= e^{-t/2}\left(\cos\left(\tfrac{\sqrt{15}}{2}t\right) + i\sin\left(\tfrac{\sqrt{15}}{2}t\right)\right)\begin{pmatrix} 4 \\ -1+i\sqrt{15} \end{pmatrix} \\
&= e^{-t/2}\begin{pmatrix} 4\cos\left(\frac{\sqrt{15}}{2}t\right) \\ -\cos\left(\frac{\sqrt{15}}{2}t\right) - \sqrt{15}\sin\left(\frac{\sqrt{15}}{2}t\right) \end{pmatrix} + \\
&\quad ie^{-t/2}\begin{pmatrix} 4\sin\left(\frac{\sqrt{15}}{2}t\right) \\ -\sin\left(\frac{\sqrt{15}}{2}t\right) + \sqrt{15}\cos\left(\frac{\sqrt{15}}{2}t\right) \end{pmatrix}.
\end{aligned}$$

By taking real and imaginary parts

$$\mathbf{Y}_{\text{re}}(t) = e^{-t/2}\begin{pmatrix} 4\cos\left(\frac{\sqrt{15}}{2}t\right) \\ -\cos\left(\frac{\sqrt{15}}{2}t\right) - \sqrt{15}\sin\left(\frac{\sqrt{15}}{2}t\right) \end{pmatrix}$$

and

$$\mathbf{Y}_{\text{im}}(t) = e^{-t/2}\begin{pmatrix} 4\sin\left(\frac{\sqrt{15}}{2}t\right) \\ -\sin\left(\frac{\sqrt{15}}{2}t\right) + \sqrt{15}\cos\left(\frac{\sqrt{15}}{2}t\right) \end{pmatrix},$$

we form the general solution $k_1\mathbf{Y}_{\text{re}}(t) + k_2\mathbf{Y}_{\text{im}}(t)$.

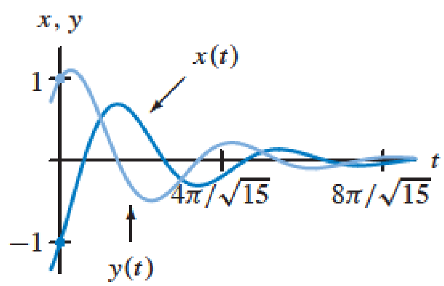
(b) To find the particular solution with the initial condition $(-1, 1)$ we set $t = 0$ in the general solution and solve for k_1 and k_2. We get

$$\begin{cases} 4k_1 = -1 \\ -k_1 + \sqrt{15}k_2 = 1, \end{cases}$$

which yields $k_1 = -1/4$ and $k_2 = 3/(\sqrt{15}4) = \sqrt{15}/20$. The desired solution is

$$\mathbf{Y}(t) = e^{-t/2} \begin{pmatrix} -\cos\left(\frac{\sqrt{15}}{2}t\right) + \frac{\sqrt{15}}{5}\sin\left(\frac{\sqrt{15}}{2}t\right) \\ \cos\left(\frac{\sqrt{15}}{2}t\right) + \frac{\sqrt{15}}{5}\sin\left(\frac{\sqrt{15}}{2}t\right) \end{pmatrix}.$$

(c)



13. **(a)** According to Exercise 7, the eigenvalues are $\lambda = (3 \pm i\sqrt{47})/2$. The eigenvalues (x, y) associated to the eigenvector $(3 + i\sqrt{47})/2$ must satisfy the equation $12y = (1 - i\sqrt{47})x$. Hence, one eigenvector is $(12, 1 - i\sqrt{47})$, and we have the complex-valued solution

$$\mathbf{Y}(t) = e^{(3+i\sqrt{47})t/2} \begin{pmatrix} 12 \\ 1 - i\sqrt{47} \end{pmatrix}$$

$$= e^{3t/2} \begin{pmatrix} 12\cos\left(\frac{\sqrt{47}}{2}t\right) \\ \cos\left(\frac{\sqrt{47}}{2}t\right) + \sqrt{47}\sin\left(\frac{\sqrt{47}}{2}t\right) \end{pmatrix} +$$

$$i e^{3t/2} \begin{pmatrix} 12\sin\left(\frac{\sqrt{47}}{2}t\right) \\ -\sqrt{47}\cos\left(\frac{\sqrt{47}}{2}t\right) + \sin\left(\frac{\sqrt{47}}{2}t\right) \end{pmatrix}.$$

Taking real and imaginary parts

$$\mathbf{Y}_{\text{re}}(t) = e^{3t/2} \begin{pmatrix} 12\cos\left(\frac{\sqrt{47}}{2}t\right) \\ \cos\left(\frac{\sqrt{47}}{2}t\right) + \sqrt{47}\sin\left(\frac{\sqrt{47}}{2}t\right) \end{pmatrix}$$

and

$$\mathbf{Y}_{\text{im}}(t) = e^{3t/2} \begin{pmatrix} 12\sin\left(\frac{\sqrt{47}}{2}t\right) \\ -\sqrt{47}\cos\left(\frac{\sqrt{47}}{2}t\right) + \sin\left(\frac{\sqrt{47}}{2}t\right) \end{pmatrix},$$

we obtain the general solution $k_1\mathbf{Y}_{\text{re}}(t) + k_2\mathbf{Y}_{\text{im}}(t)$.

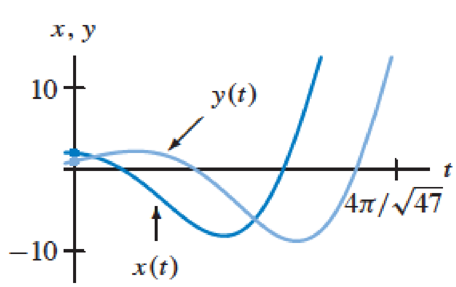
(b) From the initial condition, we have

$$k_1 \begin{pmatrix} 12 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ -\sqrt{47} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Thus, $k_1 = 1/6$ and $k_2 = -5/(6\sqrt{47})$, and the desired solution is

$$\mathbf{Y}(t) = e^{3t/2} \begin{pmatrix} 2\cos\left(\frac{\sqrt{47}}{2}t\right) - \frac{10}{\sqrt{47}}\sin\left(\frac{\sqrt{47}}{2}t\right) \\ \cos\left(\frac{\sqrt{47}}{2}t\right) + \frac{7}{\sqrt{47}}\sin\left(\frac{\sqrt{47}}{2}t\right) \end{pmatrix}.$$

(c)



14. **(a)** The characteristic polynomial is

$$(1-\lambda)(2-\lambda)+12=\lambda^2-3\lambda+14,$$

so the eigenvalues are $\lambda = 3/2 \pm i\sqrt{47}/2$. The eigenvectors (x, y) associated to the eigenvalue $(3+i\sqrt{47})/2$ must satisfy the equation $8y = (1+i\sqrt{47})x$. Hence, $(8, 1+i\sqrt{47})$ is an eigenvector, and we obtain the complex-valued solution

$$\begin{aligned}
\mathbf{Y}(t) &= e^{(3+i\sqrt{47})t/2}\begin{pmatrix} 8 \\ 1+i\sqrt{47} \end{pmatrix} \\
&= e^{3t/2}\left(\cos\left(\frac{\sqrt{47}}{2}t\right)+i\sin\left(\frac{\sqrt{47}}{2}t\right)\right)\begin{pmatrix} 8 \\ 1+i\sqrt{47} \end{pmatrix} \\
&= e^{3t/2}\begin{pmatrix} 8\cos\left(\frac{\sqrt{47}}{2}t\right) \\ \cos\left(\frac{\sqrt{47}}{2}t\right)-\sqrt{47}\sin\left(\frac{\sqrt{47}}{2}t\right) \end{pmatrix} + \\
&\qquad i e^{3t/2}\begin{pmatrix} 8\sin\left(\frac{\sqrt{47}}{2}t\right) \\ \sin\left(\frac{\sqrt{47}}{2}t\right)+\sqrt{47}\cos\left(\frac{\sqrt{47}}{2}t\right) \end{pmatrix}.
\end{aligned}$$

Taking real and imaginary parts

$$\mathbf{Y}_{\mathrm{re}}(t) = e^{3t/2}\begin{pmatrix} 8\cos\left(\frac{\sqrt{47}}{2}t\right) \\ \cos\left(\frac{\sqrt{47}}{2}t\right)-\sqrt{47}\sin\left(\frac{\sqrt{47}}{2}t\right) \end{pmatrix}$$

and

$$\mathbf{Y}_{\mathrm{im}}(t) = e^{3t/2}\begin{pmatrix} 8\sin\left(\frac{\sqrt{47}}{2}t\right) \\ \sin\left(\frac{\sqrt{47}}{2}t\right)+\sqrt{47}\cos\left(\frac{\sqrt{47}}{2}t\right) \end{pmatrix},$$

we obtain the general solution $k_1\mathbf{Y}_{\mathrm{re}}(t)+k_2\mathbf{Y}_{\mathrm{im}}(t)$.

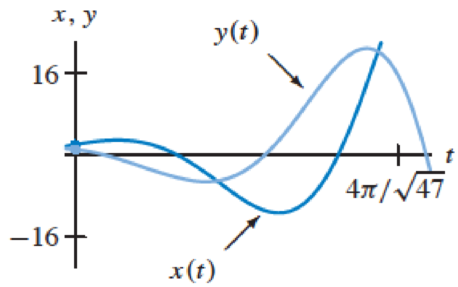
(b) To find the particular solution, we solve

$$\begin{cases} 8k_1 = 1 \\ k_1 + \sqrt{47}k_2 = -1 \end{cases}$$

and obtain $k_1 = 1/8$ and $k_1 = -9/(8\sqrt{47})$. The desired solution is

$$\mathbf{Y}(t) = e^{3t/2} \begin{pmatrix} \cos\left(\frac{\sqrt{47}}{2}t\right) - \frac{9}{\sqrt{47}}\sin\left(\frac{\sqrt{47}}{2}t\right) \\ -\cos\left(\frac{\sqrt{47}}{2}t\right) - \frac{7}{\sqrt{47}}\sin\left(\frac{\sqrt{47}}{2}t\right) \end{pmatrix}.$$

(c)



15. **(a)** In the case of complex eigenvalues, the function $x(t)$ oscillates about $x = 0$ with constant period, and the amplitude of successive oscillations is either increasing, decreasing, or constant depending on the sign of the real part of the eigenvalue. The graphs that satisfy these properties are (iv) and (v).

(b) For (iv), the natural period is approximately 1.5, and since the amplitude tends toward zero as t increases, the origin is a sink. For (v), the natural period is approximately 1.25, and since the amplitude increases as t increases, the origin is a source.

(c) (i) The time between successive zeros is not constant.
(ii) Oscillation stops at some t.
(iii) The amplitude is not monotonically decreasing or increasing.
(vi) Oscillation starts at some t. There was no prior oscillation.

16. The characteristic polynomial is

$$(a-\lambda)(a-\lambda) + b^2 = \lambda^2 - 2a\lambda + (a^2+b^2),$$

so the eigenvalues are

$$\lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2+b^2)}}{2} = a \pm \frac{\sqrt{-4b^2}}{2} = a \pm \sqrt{-b^2}.$$

Since $b^2 > 0$, the eigenvalues are complex. In fact, they are $a \pm bi$.

17. We know that $\lambda_1 = \alpha + i\beta$ satisfies the equation $\lambda_1^2 + a\lambda_1 + b = 0$. Therefore, if we take the complex conjugate all of the terms in this equation, we obtain

$$(\alpha - i\beta)^2 + a(\alpha - i\beta) + b = 0,$$

since a and b are real. The complex conjugate of λ_1 is $\lambda_2 = \alpha - i\beta$, and we have

$$\lambda_2^2 + a\lambda_2 + b = 0.$$

Therefore, λ_2 is also a root.

18. Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If (x_0, y_0) is an eigenvector associated to the eigenvalue $\alpha + i\beta$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} (\alpha + i\beta)x_0 \\ (\alpha + i\beta)y_0 \end{pmatrix}.$$

Then $ax_0 + by_0 = (\alpha + i\beta)x_0$, which is equivalent to

$$y_0 = \frac{a - \alpha + i\beta}{b} x_0.$$

Suppose x_0 is real and nonzero, then the imaginary part of y_0 is $\beta x_0 / b$. Since $\beta \neq 0$, the imaginary part of y_0 must be nonzero. (Note: If $b = 0$, then the eigenvalues are a and d. In other words, they are not complex, so the hypothesis of the exercise is not satisfied).

19. Suppose $\mathbf{Y}_2 = k\mathbf{Y}_1$ for some constant k. Then, $\mathbf{Y}_0 = (1 + ik)\mathbf{Y}_1$. Since $\mathbf{A}\mathbf{Y}_0 = \lambda\mathbf{Y}_0$, we have

$$(1 + ik)\mathbf{A}\mathbf{Y}_1 = \lambda(1 + ik)\mathbf{Y}_1.$$

Thus, $\mathbf{A}\mathbf{Y}_1 = \lambda\mathbf{Y}_1$. Now note that the left-hand side, $\mathbf{A}\mathbf{Y}_1$, is a real vector. However, since λ is complex and $\mathbf{Y}_1$ is real, the right-hand side is complex (that is, it has a nonzero imaginary part). Thus, we have a contradiction, and $\mathbf{Y}_1$ and $\mathbf{Y}_2$ must be linearly independent.

20. If $\mathbf{A}\mathbf{Y}_0 = \lambda\mathbf{Y}_0$, then we can take complex conjugates of both sides to obtain $\overline{\mathbf{A}\mathbf{Y}_0} = \overline{\lambda\mathbf{Y}_0}$ (where the complex conjugate of a vector or matrix is the complex conjugate of its entries). But $\overline{\mathbf{A}\mathbf{Y}_0} = \overline{\mathbf{A}}\overline{\mathbf{Y}_0} = \mathbf{A}\overline{\mathbf{Y}_0}$ because $\mathbf{A}$ is real. Also, $\overline{\lambda\mathbf{Y}_0} = \overline{\lambda}\,\overline{\mathbf{Y}_0}$. Hence, $\mathbf{A}\overline{\mathbf{Y}_0} = \overline{\lambda}\overline{\mathbf{Y}_0}$. In other words, $\overline{\lambda}$ is an eigenvalue of $\mathbf{A}$ with eigenvector $\overline{\mathbf{Y}_0}$.

21. **(a)** The factor $e^{-\alpha t}$ is positive for all t. Hence, the zeros of $x(t)$ are exactly the zeros of $\sin \beta t$. Suppose t_1 and t_2 are successive zeros (that is, $t_1 < t_2$, $x(t_1) = x(t_2) = 0$, and $x(t) \neq 0$ for $t_1 < t < t_2$), then $\beta t_2 - \beta t_1 = \pi$. In other words, $t_2 - t_1 = \pi/\beta$.

(b) By the nature of sine function, local maxima and local minima appear alternately. Therefore, we look for t_1 and t_2 such that $x'(t_1) = x'(t_2) = 0$ and $x'(t) \neq 0$ for $t_1 < t < t_2$. From

$$x'(t) = e^{-\alpha t}(-\alpha \sin \beta t + \beta \cos \beta t) = 0,$$

we know that $\tan \beta t = \beta/\alpha$ if t is corresponds to a local extremum. Since the tangent function is periodic with period π, $\beta(t_2 - t_1) = \pi$. Hence, $t_2 - t_1 = \pi/\beta$. Note that the distance between a local minimum and the following local maximum of $x(t)$ is constant over t.

(c) From part (b), we know that the distance between the first local maximum and the first local minimum is π/β and the distance between the first local minimum and the second local maximum is π/β. Therefore, the distance between the first two local maxima of $x(t)$ is $2\pi/\beta$.

(d) From part (b), we know that the first local maximum of $x(t)$ occurs at $t = (\arctan(\beta/\alpha))/\beta$.

22. Consider the point in the plane determined by the coordinates (k_1, k_2), and let ϕ be an angle such that $K \cos \phi = k_1$ and $K \sin \phi = k_2$. (Such an angle exists since $(K \cos \phi, K \sin \phi)$ parameterizes the circle through (k_1, k_2) centered at the origin. In fact, there are infinitely many such ϕ, all differing by integer multiples of 2π.) Then

$$\begin{aligned} x(t) &= k_1 \cos \beta t + k_2 \sin \beta t \\ &= K \cos \phi \cos \beta t + K \sin \phi \sin \beta t \\ &= K \cos(\beta t - \phi). \end{aligned}$$

The last equality comes from the trigonometric identity for the cosine of the difference of two angles.

23. **(a)** The corresponding first-order system is

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -qy - pv.$$

(b) The characteristic polynomial is

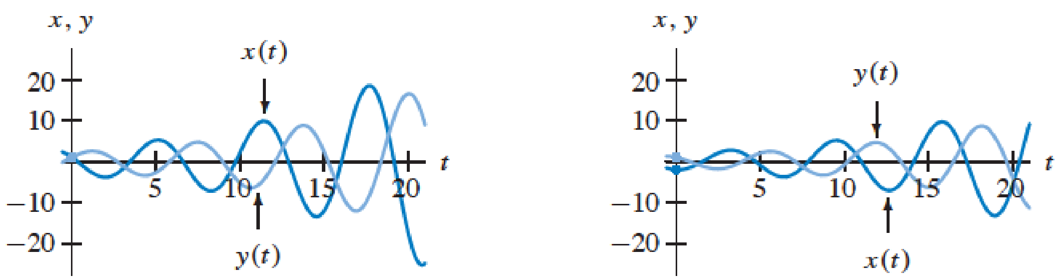
$$(-\lambda)(-p - \lambda) + q = \lambda^2 + p\lambda + q,$$

so the eigenvalues are $\lambda = (-p \pm \sqrt{p^2 - 4q})/2$. Hence, the eigenvalues are complex if and only if $p^2 < 4q$. Note that q must be positive for this condition to be satisfied.

(c) In order to have a spiral sink, we must have $p^2 < 4q$ (to make the eigenvalues complex) and $p > 0$ (to make the real part of the eigenvalues negative). In other words, the origin is a spiral sink if and only if $q > 0$ and $0 < p < 2\sqrt{q}$. The origin is a center if and only if $q > 0$ and $p = 0$. Finally, the origin is a spiral source if and only if $q > 0$ and $-2\sqrt{q} < p < 0$.

(d) The vector field at $(1, 0)$ is $(0, -q)$. Hence, if $q > 0$, then the vector field points down along the entire y-axis, and the solution curves spiral about the origin in a clockwise fashion. Note that q must be positive for the eigenvalues to be complex, so the solution curves always spiral about the origin in a clockwise fashion as long as the eigenvalues are complex.

24. Note that the graphs have the same period and exponential rate of growth.



25. There is no spiral saddle because a linear saddle is a linear system where some solutions approach the origin and some move away. If one solution spirals toward (or away from) the origin, then we can multiply that solution by any constant, scaling it so that it goes through any point in the plane. This scaled solution is still a solution of the system (recall the Linearity Principle), so every solution spirals in the same way, either toward or away from the origin.

26. The eigenvalues are $\pm i$. Using the usual method to find eigenvectors, we see that the eigenvectors corresponding to the eigenvalue i satisfy the equation $10y = (3+i)x$. We use the eigenvector $\mathbf{V}_0 = (10, 3+i)$ to determine the general solution. It is

$$\mathbf{Y}(t) = k_1 \begin{pmatrix} 10\cos t \\ 3\cos t - \sin t \end{pmatrix} + k_2 \begin{pmatrix} 10\sin t \\ \cos t + 3\sin t \end{pmatrix}.$$

In terms of components, we have

$$x(t) = 10k_1 \cos t + 10k_2 \sin t$$
$$y(t) = (3k_1 + k_2)\cos t + (3k_2 - k_1)\sin t.$$

To show that the solution curves are ellipses, we need to find an "elliptical" relationship that $x(t)$ and $y(t)$ satisfy. In this case, it turns out that

$$[x(t)]^2 - 6x(t)\,y(t) + 10[y(t)]^2 = 10(k_1^2 + k_2^2).$$

In particular, the value of $x^2 - 6xy + 10y^2$ does not depend on t. It only depends on k_1 and k_2, which are, in turn, determined by the initial condition. It is an exercise in analytic geometry to show that the curves that satisfy

$$x^2 - 6xy + 10y^2 = K$$

are ellipses for any positive constant K.

You may wonder where $x^2 - 6xy + 10y^2$ comes from. See the technique for constructing Hamiltonian functions described in Section 5.3.