1. The characteristic polynomial is
$$s^2 - 6s - 7,$$
so the eigenvalues are $s = -1$ and $s = 7$. Hence, the general solution is
$$y(t) = k_1 e^{-t} + k_2 e^{7t}.$$

3. The characteristic polynomial is
$$s^2 + 6s + 9,$$
so $s = -3$ is a repeated eigenvalue. Hence, the general solution is
$$y(t) = k_1 e^{-3t} + k_2 t e^{-3t}.$$

5. The characteristic polynomial is
$$s^2 + 8s + 25,$$
so the complex eigenvalues are $s = -4 \pm 3i$. Hence, the general solution is
$$y(t) = k_1 e^{-4t} \cos 3t + k_2 e^{-4t} \sin 3t.$$

7. The characteristic polynomial is
$$s^2 + 2s - 3,$$
so the eigenvalues are $s = 1$ and $s = -3$. Hence, the general solution is
$$y(t) = k_1 e^{t} + k_2 e^{-3t},$$
and we have
$$y'(t) = k_1 e^{t} - 3k_2 e^{-3t}.$$
From the initial conditions, we obtain the simultaneous equations
$$\begin{cases} k_1 + k_2 = 6 \\ k_1 - 3k_2 = -2. \end{cases}$$
Solving for k_1 and k_2 yields $k_1 = 4$ and $k_2 = 2$. Hence, the solution to our initial-value problem is $y(t) = 4e^{t} + 2e^{-3t}$.

13. **(a)** The resulting second-order equation is

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 7y = 0,$$

and the corresponding system is

$$\begin{aligned}\frac{dy}{dt} &= v\\ \frac{dv}{dt} &= -7y - 8v.\end{aligned}$$

(b) Recall that we can read off the characteristic equation of the second-order equation straight from the equation without having to revert to the corresponding system. We obtain

$$\lambda^2 + 8\lambda + 7 = 0.$$

Therefore, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -7$.

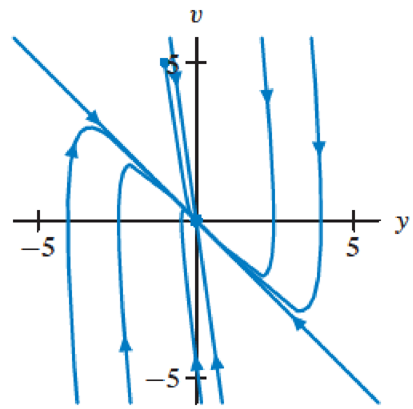
To find the eigenvectors associated to the eigenvalue λ_1, we solve the simultaneous system of equations

$$\begin{cases} v = -y \\ -7y - 8v = -v. \end{cases}$$

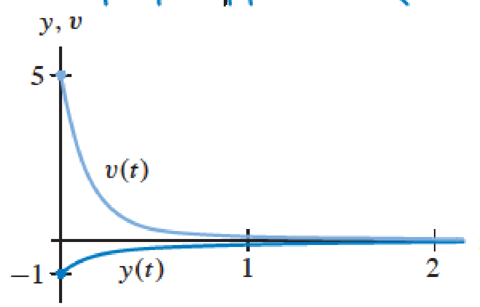
From the first equation, we immediately see that the eigenvectors associated to this eigenvalue must satisfy $v = -y$. Similarly, the eigenvectors associated to the eigenvalue $\lambda_2 = -7$ must satisfy the equation $v = -7y$.

(c) Since the eigenvalues are real and negative, the equilibrium point at the origin is a sink, and the system is overdamped.

(d) We know that all solution curves approach the origin as $t \to \infty$ and, with the exception of those whose initial conditions lie on the line $v = -7y$, these solution curves approach the origin tangent to the line $v = -y$.



(e) From the phase portrait, we see that $y(t)$ increases monotonically toward 0 as $t \to \infty$. Also, $v(t)$ decreases monotonically toward 0. It is useful to remember that $v = dy/dt$.



15. **(a)** The resulting second-order equation is

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = 0,$$

and the corresponding system is

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -5y - 4v. \end{aligned}$$

(b) Recall that we can read off the characteristic equation of the second-order equation straight from the equation without having to revert to the corresponding system. We obtain

$$\lambda^2 + 4\lambda + 5 = 0.$$

Therefore, the eigenvalues are $\lambda_1 = -2 + i$ and $\lambda_2 = -2 - i$.

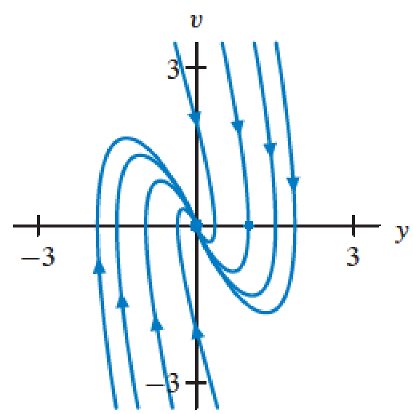
To find the eigenvectors associated to the eigenvalue λ_1, we solve the simultaneous system of equations

$$\begin{cases} v = (-2+i)y \\ -5y - 4v = (-2+i)v. \end{cases}$$

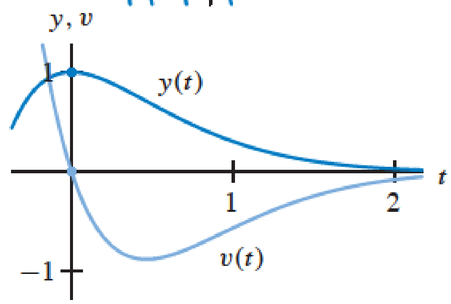
From the first equation, we immediately see that the eigenvectors associated to this eigenvalue must satisfy $v = (-2+i)y$. Similarly, the eigenvectors associated to the eigenvalue $\lambda_2 = -2-i$ must satisfy the equation $v = (-2-i)y$.

(c) Since the eigenvalues are complex with negative real part, the equilibrium point at the origin is a spiral sink, and the system is underdamped.

(d) All solutions tend to the origin spiralling in the clockwise direction with period 2π. Admittedly, it is difficult to see these oscillations in the picture.



(e) The graph of $y(t)$ initially decreases then oscillates with decreasing amplitude as it tends to 0. Similarly, $v(t)$ initially decreases and becomes negative, then oscillates with decreasing amplitude as it tends to 0.



17. **(a)** The resulting second-order equation is

$$2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = 0,$$

and the corresponding system is

$$\begin{aligned}\frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -\frac{1}{2}y - \frac{3}{2}v.\end{aligned}$$

(b) Recall that we can read off the characteristic equation of the second-order equation straight from the equation without having to revert to the corresponding system. We obtain

$$2\lambda^2 + 3\lambda + 1 = 0.$$

Therefore, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -1/2$.

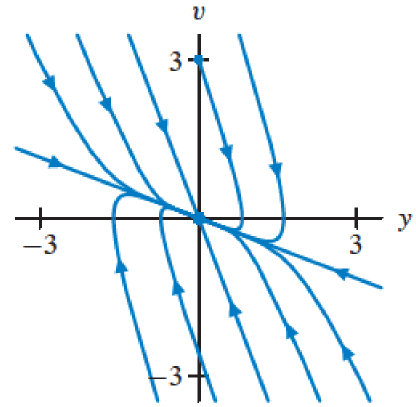
To find the eigenvectors associated to the eigenvalue λ_1, we solve the simultaneous system of equations

$$\left\{\begin{aligned} v &= -y \\ -\tfrac{1}{2}y - \tfrac{3}{2}v &= -v. \end{aligned}\right.$$

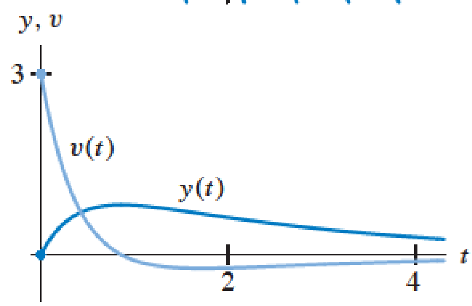
From the first equation, we immediately see that the eigenvectors associated to this eigenvalue must satisfy $v = -y$. Similarly, the eigenvectors associated to the eigenvalue $\lambda_2 = -1/2$ must satisfy the equation $v = -y/2$.

(c) Since the eigenvalues are real and negative, the equilibrium point at the origin is a sink, and the system is overdamped.

(d) We know that all solution curves approach the origin as $t \to \infty$ and, with the exception of those whose initial conditions lie on the line $v = -y$, these solution curves approach the origin tangent to the line $v = -y/2$.



(e) According to the phase plane, $y(t)$ increases initially. Eventually it reaches a maximum value. Then it approaches 0 as $t \to \infty$. Also, $v(t)$ decreases, becomes negative, and then approaches 0 from below. While sketching these graphs, it is useful to remember that $v = dy/dt$.



19. **(a)** The resulting second-order equation is

$$2\frac{d^2y}{dt^2} + 3y = 0,$$

and the corresponding system is

$$\begin{aligned}\frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -\frac{3}{2}y.\end{aligned}$$

(b) Recall that we can read off the characteristic equation of the second-order equation straight from the equation without having to revert to the corresponding system. We obtain

$$2\lambda^2 + 3 = 0.$$

Therefore, we have pure imaginary eigenvalues, $\lambda = \pm i\sqrt{3/2}$.

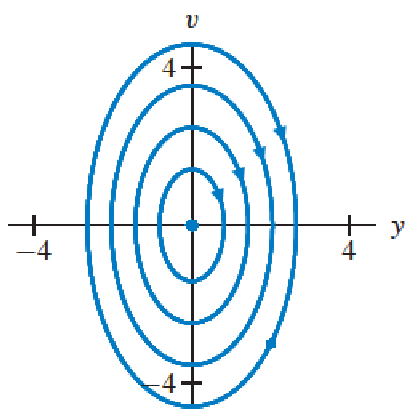
To find the eigenvectors associated to the eigenvalue $\lambda = i\sqrt{3/2}$, we solve the simultaneous system of equations

$$\begin{cases} v = i\sqrt{\frac{3}{2}}\,y \\ -\frac{3}{2}y = i\sqrt{\frac{3}{2}}\,v. \end{cases}$$

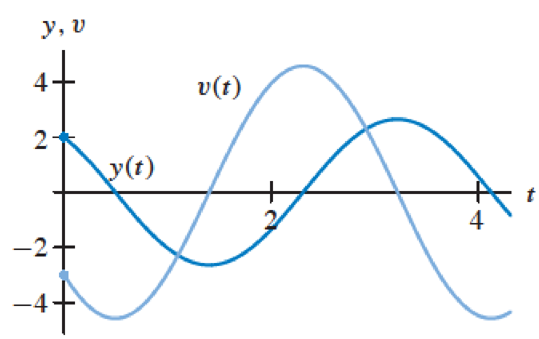
From the first equation, we immediately see that the eigenvectors associated to this eigenvalue must satisfy $v = i\sqrt{3/2}\,y$. Similarly, the eigenvectors associated to the eigenvalue $\lambda = -i\sqrt{3/2}$ must satisfy the equation $v = -i\sqrt{3/2}\,y$.

(c) Since the eigenvalues are pure imaginary, the system is undamped. (Of course, we already knew this because $b = 0$.) The natural period is $2\pi/\sqrt{3/2} = 4\pi/\sqrt{6}$.

(d) Since the eigenvalues are pure imaginary, we know that the solution curves are ellipses. At the point $(1, 0)$, $d\mathbf{Y}/dt = (0, -3/2)$. Therefore, we know that the oscillation is clockwise.



(e)



21. **(a)** The second-order equation is

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 7y = 0,$$

so the characteristic equation is

$$s^2 + 8s + 7 = 0.$$

The roots are $s = -7$ and $s = -1$. The general solution is

$$y(t) = k_1 e^{-7t} + k_2 e^{-t}.$$

(b) To find the particular solution we compute

$$v(t) = -7k_1 e^{-7t} - k_2 e^{-t}.$$

The particular solution satisfies

$$\begin{cases} -1 = y(0) = k_1 + k_2 \\ 5 = v(0) = -7k_1 - k_2. \end{cases}$$

The first equation yields $k_1 = -k_2 - 1$. Substituting into the second we obtain $5 = 6k_2 + 7$, which implies $k_2 = -1/3$. The first equation then yields $k_1 = -2/3$. The particular solution is

$$y(t) = -\tfrac{2}{3}e^{-7t} - \tfrac{1}{3}e^{-t}.$$

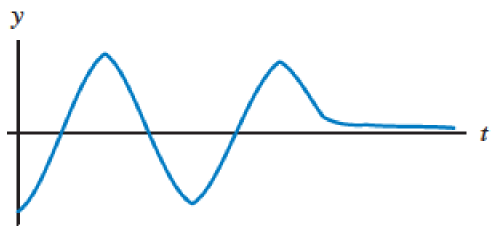
(c) The $y(t)$- and $v(t)$-graphs are displayed in the solution of Exercise 13.

29. Note: We assume that m, k and b are nonnegative—the physically relevant case. All references to graphs and phase portraits are from Sections 3.5 and 3.6.

Table 3.1
Possible harmonic oscillators.

name	eigenvalues	parameters	decay rate	phase portrait and graphs
undamped	pure imaginary	$b = 0$	no decay	Figure 3.41
underdamped	complex with negative real part	$b^2 - 4mk < 0$	$e^{-bt/(2m)}$	Figure 3.42
critically damped	only one eigenvalue	$b^2 - 4mk = 0$	$e^{-bt/(2m)}$	Figure 3.34
overdamped	two negative real	$b^2 - 4mk > 0$	$e^{\lambda t}$ where $\lambda = \dfrac{-b + \sqrt{b^2 - 4mk}}{2m}$	Figures 3.43–3.45 and Exercise 13

36. **(a)**



(b) Using the model of a harmonic oscillator for the suspension system, the corresponding system has either real or complex eigenvalues. If it has complex eigenvalues, then solutions spiral in the phase plane and oscillations of $y(t)$ continue for all time. If there are real eigenvalues, then solutions do not spiral, and in fact, they cannot cross the v-axis (where $y = 0$) more than once. Hence, the behavior described is impossible for a harmonic oscillator.

(c) There is room for disagreement in this answer. One reasonable choice is an oscillator with complex eigenvalues and some damping so that the system does oscillate, but the amplitude of the oscillations decays sufficiently rapidly so that only the first two "bounces" are of significant size.