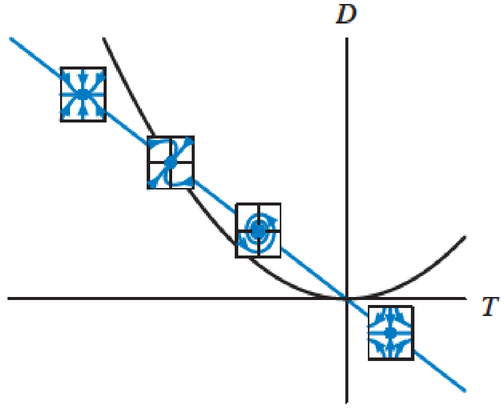
1.

Table 3.2

Possibilities for linear systems

type	condition on λ	examples
sink	$\lambda_1 < \lambda_2 < 0$	Sec. 3.7, Fig. 3.52
saddle	$\lambda_1 < 0 < \lambda_2$	Sec. 3.3, Fig. 3.12–3.14
source	$0 < \lambda_1 < \lambda_2$	Sec. 3.3, Fig. 3.19
spiral sink	$\lambda = \alpha \pm i\beta, \alpha < 0, \beta \neq 0$	Sec. 3.1, Fig. 3.2 and 3.4
spiral source	$\lambda = \alpha \pm i\beta, \alpha > 0, \beta \neq 0$	Sec. 3.4, Fig. 3.29–3.30
center	$\lambda_1 = \pm i\beta, \beta \neq 0$	Sec. 3.1, Fig. 3.1 and 3.3
		Sec. 3.4, Fig. 3.28
sink	$\lambda_1 = \lambda_2 < 0$	Sec. 3.5, Fig. 3.35–3.36
(special case)	One line of eigenvectors	
source	$0 < \lambda_1 = \lambda_2$	Sec. 3.5, Ex. 2
(special case)	One line of eigenvectors	
sink	$\lambda_1 = \lambda_2 < 0$	Sec. 3.5, Ex. 23
(special case)	Every vector is eigenvector	
source	$0 < \lambda_1 = \lambda_2$	Sec. 3.5, Ex. 23
(special case)	Every vector is eigenvector	
no name	$\lambda_1 < \lambda_2 = 0$	Sec. 3.5, Fig. 3.39–3.40
no name	$0 = \lambda_1 < \lambda_2$	Sec. 3.5, Ex. 19
no name	$\lambda_1 = \lambda_2 = 0$	Sec. 3.5, Ex. 21
	One line of eigenvectors	
no name	$\lambda_1 = \lambda_2 = 0$	entire plane of equilibrium points
	Every vector is an eigenvector	

3. **(a)**

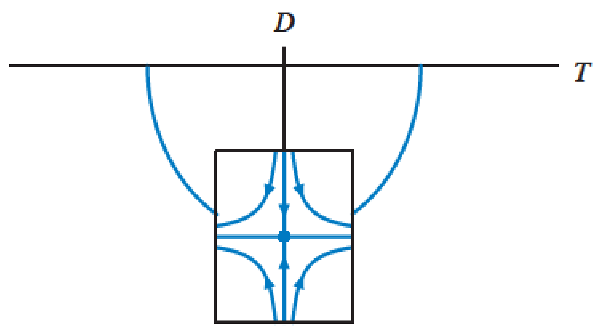


(b) The trace T is $2a$, and the determinant D is $-a$. Therefore, the curve in the trace-determinant plane is $D = -T/2$. This line crosses the parabola $T^2 - 4D = 0$ at two points—at $(T, D) = (0, 0)$ if $a = 0$ and at $(T, D) = (-2, 1)$ if $a = -1$.

The portion of the line for which $a < -1$ corresponds to a positive determinant and a negative trace such that $T^2 - 4D > 0$. The corresponding phase portraits are real sinks. If $a = -1$, we have a sink with repeated eigenvalues. If $-1 < a < 0$, we have complex eigenvalues with negative real parts. Therefore, the phase portraits are spiral sinks. If $a = 0$, we have a degenerate case with an entire line of equilibrium points. Finally, if $a > 0$, the corresponding portion of the line is below the T-axis, and the phase portraits are saddles.

(c) Bifurcations occur at $a = -1$, where we have a sink with repeated eigenvalues, and at $a = 0$, where we have zero as a repeated eigenvalue. For $a = 0$, the y-axis is entirely composed of equilibrium points.

5. **(a)**

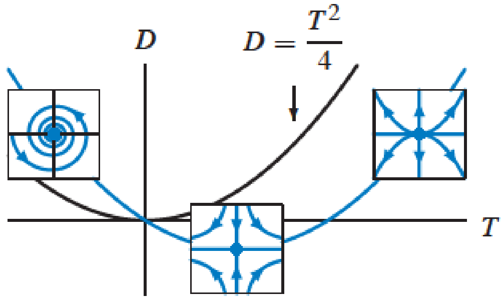


(b) The curve in the trace-determinant plane is the portion of the unit circle centered at 0 that lies in the half-plane $y \leq 0$.

A glance at the trace-determinant plane shows that for $-1 < a < 1$, we have a saddle. If $a = 1$, the eigenvalues are 0 and 1. If $a = -1$, the eigenvalues are 0 and -1.

(c) Bifurcations occur only at $a = \pm 1$. For these two special values of a, we have a line of equilibrium points. The nonzero equilibrium points disappear if $-1 < a < 1$.

7. **(a)**



(b) The trace T is $2a$, and the determinant D is $a^2 - a$. Therefore, the curve in the trace-determinant plane is

$$\begin{aligned} D &= a^2 - a \\ &= \left(\frac{T}{2}\right)^2 - \frac{T}{2} \\ &= \frac{T^2}{4} - \frac{T}{2}. \end{aligned}$$

This curve is a parabola. It meets the repeated-eigenvalue parabola (the parabola $D = T^2/4$) if

$$\frac{T^2}{4} - \frac{T}{2} = \frac{T^2}{4}.$$

Solving this equation yields $T = 0$, which corresponds to $a = 0$.

This curve also meets the T-axis (the line $D = 0$) if

$$\frac{T^2}{4} - \frac{T}{2} = 0,$$

so if $T = 0$ or $T = 2$, then $D = 0$.

From the location of the parabola $D = T^2/4 - T/2$ in the trace-determinant plane, we see that the phase portrait is a spiral sink if $a < 0$ since $T < 0$, a saddle if $0 < a < 1$ since $0 < T < 2$, and a source with distinct real eigenvalues if $a > 1$ since $T > 2$.

(c) Bifurcations occur at $a = 0$, where we have repeated zero eigenvalues, and at $a = 1$, where we have a single zero eigenvalue.

9. The eigenvalues are roots of the equation $\lambda^2 - 2a\lambda + a^2 - b^2 = 0$. These roots are

$$a \pm \sqrt{b^2} = a \pm |b|.$$

So we have a repeated zero eigenvalue if $a = b = 0$.

If $a = \pm b$, then one of the eigenvalues is 0, and as long as $a \neq 0$ (so $b \neq 0$), the other eigenvalue is nonzero.

The eigenvalues are repeated (both equal to a) if $b = 0$. The eigenvalues are never complex since $\sqrt{b^2} \geq 0$.

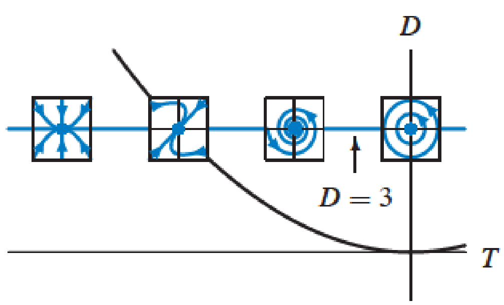
If $a > |b|$, then $a \pm |b| > 0$, so we have a source with real eigenvalues. If $a < 0$ and $-a > |b|$, then $a \pm |b| < 0$, so we have a sink with real eigenvalues. In all other cases we have a saddle.

11. **(a)** This second-order equation is equivalent to the system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -3y - bv.$$

Therefore, $T = -b$ and $D = 3$. So the corresponding curve in the trace-determinant plane is $D = 3$.

(b)



(c) The line $D = 3$ in the trace-determinant plane crosses the repeated-eigenvalue parabola $D = T^2/4$ if $b^2 = 12$, which implies that $b = 2\sqrt{3}$ since b is a nonnegative parameter. If $b = 0$, we have pure imaginary eigenvalues—the undamped case. If $0 < b < 2\sqrt{3}$, the eigenvalues are complex with a negative real part—the underdamped case. If $b = 2\sqrt{3}$, the eigenvalues are repeated and negative—the critically damped case. Finally, if $b > 2\sqrt{3}$, the eigenvalues are real and negative—the overdamped case.