1. The linearizations of systems (i) and (iii) are both

$$\frac{dx}{dt} = 2x + y$$
$$\frac{dy}{dt} = -y,$$

so these two systems have the same "local picture" near $(0, 0)$. This system has eigenvalues 2 and -1; hence, $(0, 0)$ is a saddle for these systems. System (ii) has linearization

$$\frac{dx}{dt} = 2x + y$$
$$\frac{dy}{dt} = y,$$

which has eigenvalues 2 and 1, hence, $(0, 0)$ is a source for this system.

3. **(a)** The linearized system is

$$\frac{dx}{dt} = -2x + y$$
$$\frac{dy}{dt} = -y.$$

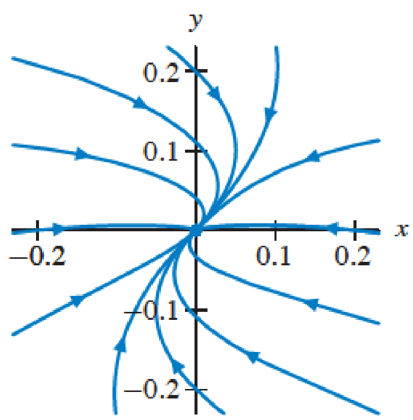
We can see this either by "dropping higher-order terms" or by computing the Jacobian matrix

$$\begin{pmatrix} -2 & 1 \\ 2x & -1 \end{pmatrix}$$

and evaluating it at $(0, 0)$.

(b) The eigenvalues of the linearized system are -2 and -1, so $(0, 0)$ is a sink.

(c) The vector $(1, 0)$ is an eigenvector for eigenvalue -2 and $(1, 1)$ is an eigenvector for the eigenvalue -1.



9. **(a)** The equilibrium points are $(0,0)$, $(0,25)$, $(100,0)$ and $(75, 12.5)$. We classify these equilibrium points by computing the Jacobian matrix, which is

$$\begin{pmatrix} 100 - 2x - 2y & -2x \\ -y & 150 - x - 12y \end{pmatrix},$$

and evaluating it at each of the equilibrium points. At $(0,0)$, the Jacobian matrix is

$$\begin{pmatrix} 100 & 0 \\ 0 & 150 \end{pmatrix},$$

and the eigenvalues are 100 and 150. So this point is a source. At $(0,25)$, the Jacobian matrix is

$$\begin{pmatrix} 50 & 0 \\ -25 & -150 \end{pmatrix},$$

and the eigenvalues are 50 and -150. Hence, this point is a saddle. At $(100,0)$, the Jacobian matrix is

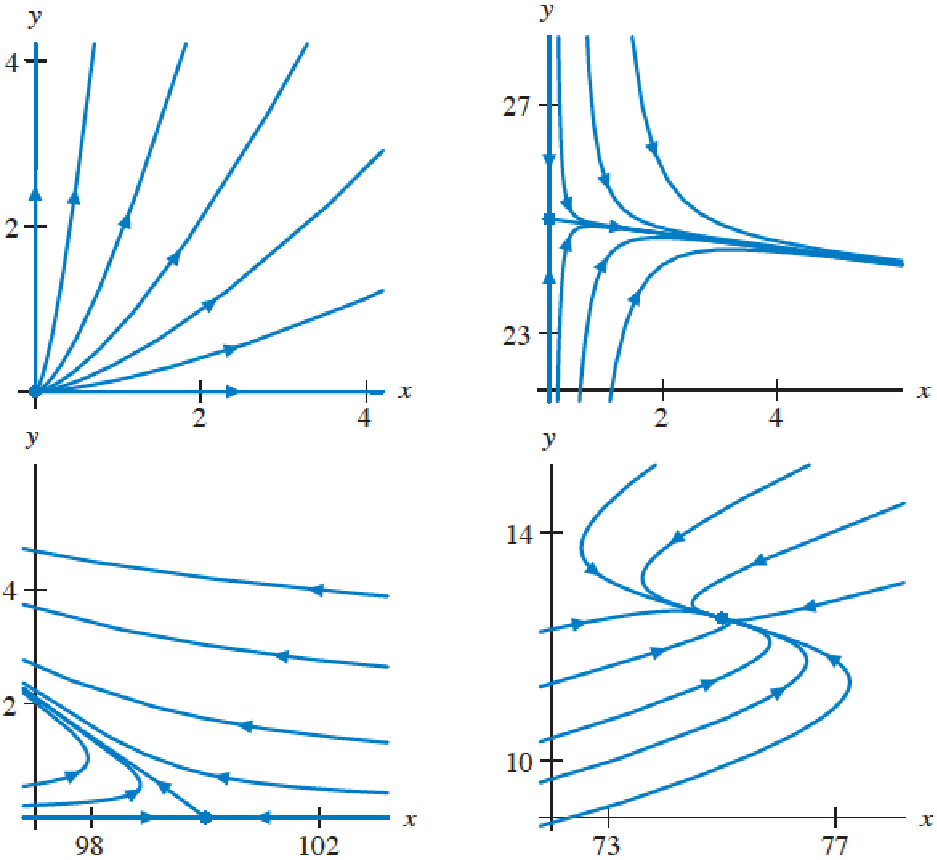
$$\begin{pmatrix} -100 & -200 \\ 0 & 50 \end{pmatrix},$$

and the eigenvalues are -100 and 50. Therefore, this point is a saddle. Finally, at $(75, 12.5)$, the Jacobian matrix is

$$\begin{pmatrix} -75 & -150 \\ -12.5 & -75 \end{pmatrix},$$

and the eigenvalues are approximately -32 and -118. So this point is a sink.

(b)



11. **(a)** The equilibrium points in the first quadrant are $(0, 0)$, $(0, 50)$ and $(40, 0)$. To classify these equilibrium points, we compute the Jacobian matrix, which is

$$\begin{pmatrix} -2x - y + 40 & -x \\ -2xy & -x^2 - 3y^2 + 2500 \end{pmatrix},$$

and we evaluate it at each of the points. At $(0, 0)$, the Jacobian matrix is

$$\begin{pmatrix} 40 & 0 \\ 0 & 2500 \end{pmatrix},$$

which has eigenvalues 40 and 2500. Therefore, $(0, 0)$ is a source. At $(0, 50)$, the Jacobian matrix is

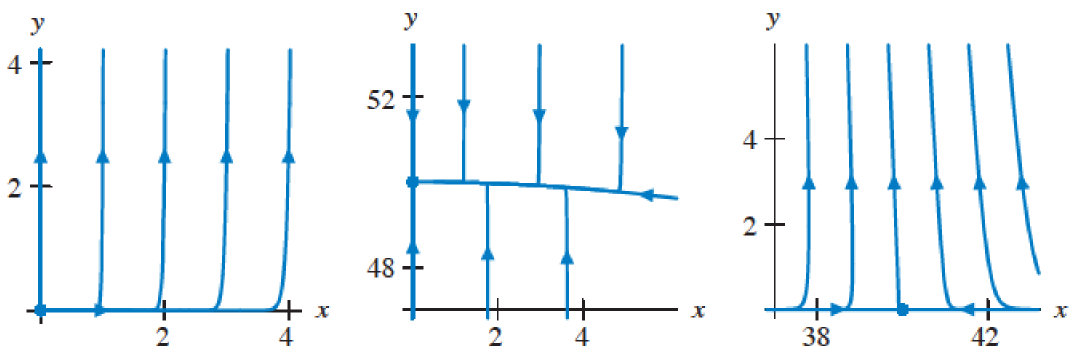
$$\begin{pmatrix} -10 & 0 \\ 0 & -5000 \end{pmatrix},$$

which has eigenvalues -10 and -5000. So $(0, 50)$ is a sink. At $(40, 0)$, the Jacobian matrix is

$$\begin{pmatrix} -40 & -40 \\ 0 & 900 \end{pmatrix},$$

which has eigenvalues -40 and 900. Hence, $(40, 0)$ is a saddle.

(b)



15. **(a)** The equilibrium points are (0, 0), (1, 1) and (2, 0). We determine the type of each of these points by computing the Jacobian, which is

$$\begin{pmatrix} 2-2x-y & -x \\ -y & 2y-x \end{pmatrix},$$

and evaluating it at the points. At (0, 0), the Jacobian is

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

which has eigenvalues 2 and 0. An eigenvector for the eigenvalue 2 is (1, 0), so solutions move away from the origin parallel to the x-axis. On the line $x = 0$, we have $dy/dt = y^2$ so solutions move upwards when $y \neq 0$. Hence, (0, 0) is a node. However, solutions near the origin in the first quadrant move away from the origin as t increases. At (1, 1), the Jacobian is

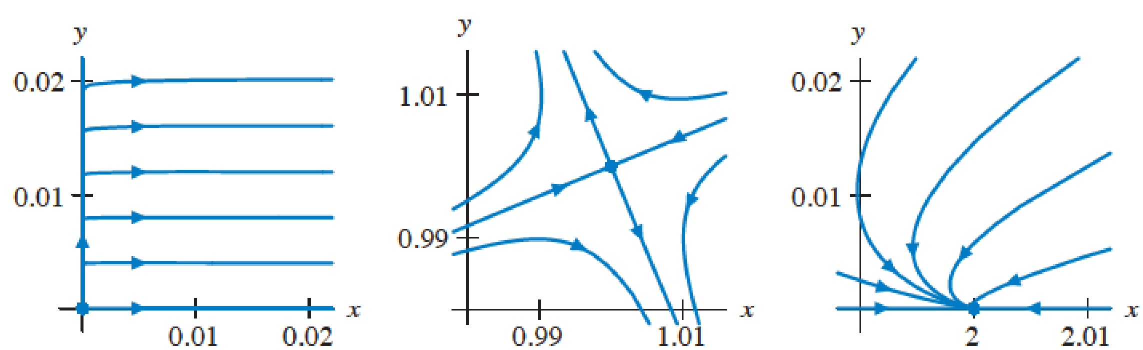
$$\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix},$$

which has eigenvalues $\pm\sqrt{2}$. So (1, 1) is a saddle. At (2, 0), the Jacobian is

$$\begin{pmatrix} -2 & -2 \\ 0 & -2 \end{pmatrix},$$

which has a double eigenvalue of -2. Therefore, (2, 0) is a sink.

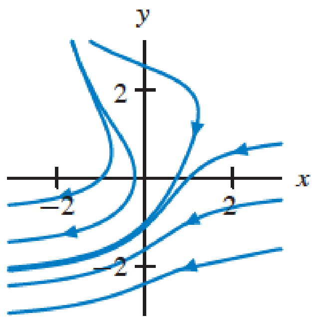
(b)



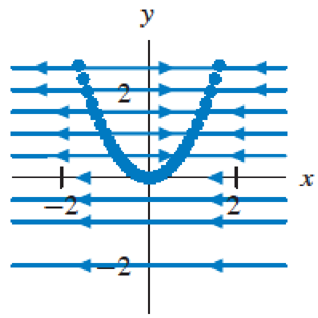
21. **(a)** The only equilibrium points occur if $a = 0$. Then all points on the curve $y = x^2$ are equilibrium points.

(b) The bifurcation occurs at $a = 0$.

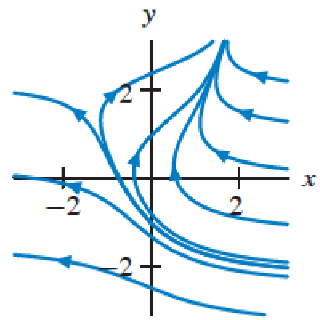
(c) If $a < 0$, all solutions decrease in the y-direction since $dy/dt < 0$. If $a > 0$, all solutions increase in the y-direction since $dy/dt > 0$. If $a = 0$, there is a curve of equilibrium points located along $y = x^2$, and all solutions move horizontally.



Phase portrait for $a < 0$



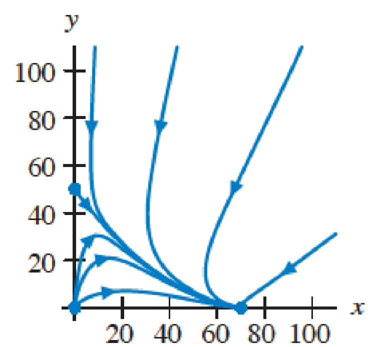
Phase portrait for $a = 0$



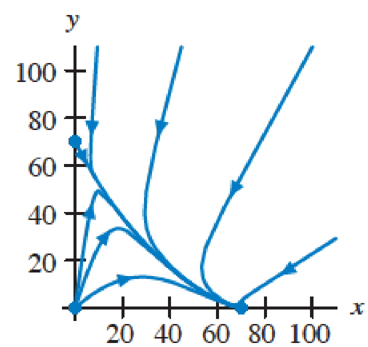
Phase portrait for $a > 0$

26. Since this is a competing species model, $a > 0$. The equilibrium points are $(0, 0)$, $(0, a)$, $(70, 0)$, and $(a - 70, 140 - a)$. If $a = 70$, the second and fourth of these points coincide. If $a = 140$, the third and fourth coincide. Hence bifurcations occur at these two a-values.

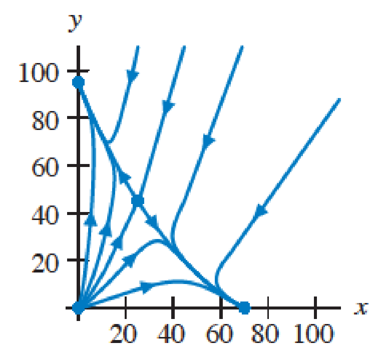
For $70 < a < 140$ there is an equilibrium point that does not lie on the axes. This equilibrium point is a saddle whose separatrices divide the first quadrant into two regions. In one region, all solutions tend to $(0, a)$ and in the other, to $(70, 0)$. If $0 < a < 70$, all solutions (not on the axes) tend to the equilibrium point at $(70, 0)$; that is, the y-species dies out. If $a > 140$, all solutions (not on the axes) tend to the equilibrium point at $(0, a)$; that is, the x-species dies out.



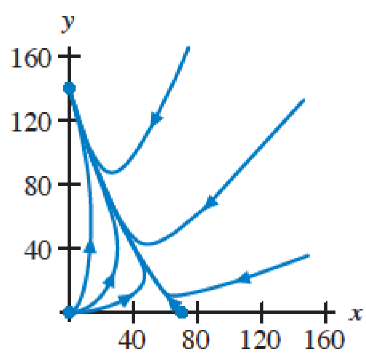
Phase portrait for $a < 70$



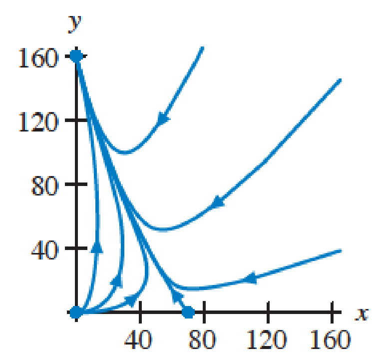
Phase portrait for $a = 70$



Phase portrait for $70 < a < 140$



Phase portrait for $a = 140$



Phase portrait for $a > 140$

27. **(a)** The fact that (0, 0) is an equilibrium point says that, if both X and Y are absent from the island, then neither will ever migrate to the island. However, it may be possible for one species to migrate if the other is already on the island.

(b) If a small population consisting solely of one of the species reproduces rapidly, then we expect both $\partial f/\partial x$ and $\partial g/\partial y$ to be positive and large at $(0, 0)$. We expect this because these partials are the coefficients of x and y in the linearization at $(0, 0)$.

(c) Since the species compete, an increase in y decreases dx/dt and an increase in x decreases dy/dt. Hence, both $\partial f/\partial y$ and $\partial g/\partial x$ are negative at (0, 0) since $\partial f/\partial y$ is the coefficient of y in the dx/dt equation and $\partial g/\partial x$ is the coefficient for x in the dy/dt equation for the linearization at the origin.

(d) Suppose the coefficient matrix of the linearized system is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

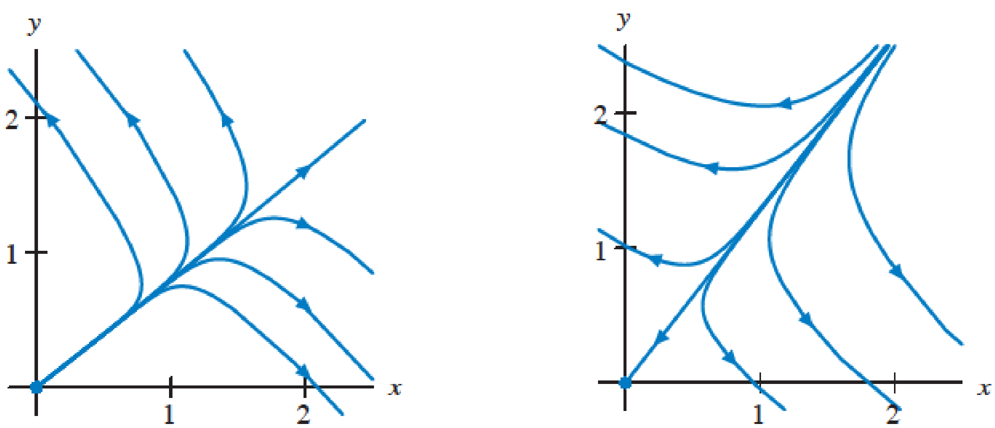
with a and d positive and large and b and c negative. The eigenvalues are

$$\frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}.$$

If b and c are near zero, then $(0, 0)$ is a source. If b and c are very negative, then $(0, 0)$ is a saddle.

It is also possible to have 0 as an eigenvalue of the linearized system in which case the linearization fails to determine the behavior of the nonlinear system near $(0, 0)$.

(e) For the linearized system, note that $dx/dt < 0$ along the positive y-axis and $dy/dt < 0$ along the positive x-axis. If the origin is a saddle, the eigenvectors for the negative eigenvalue must be in the first and third quadrants, and a typical solution near the origin starting in the first quadrant has one of the species going extinct. If the origin is a source, then a typical solution near the origin has one or the other of the species going extinct except for one curve of solutions in the first quadrant.



28. **(a)** At $(0, 0)$, $\partial f/\partial x$ and $\partial g/\partial y$ are positive and small.

(b) At $(0, 0)$, $\partial f/\partial y$ and $\partial g/\partial x$ are negative and large in absolute value.

(c) With these assumptions, the Jacobian matrix is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

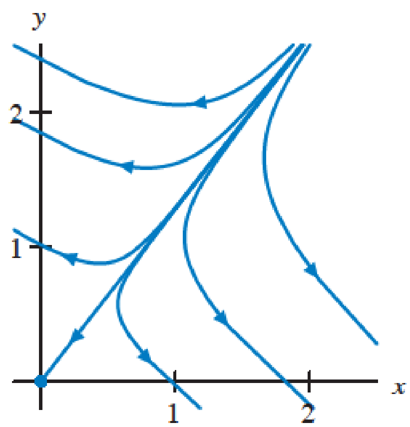
where a and d are small and positive, but b and c are negative with much larger absolute value. Since the eigenvalues of this matrix are given by

$$\frac{a + d \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2}$$

and since $(a + d)^2 > 0$ and $ad - bc < 0$, the term inside the square root is positive. Thus both eigenvalues are real.

The term $a + d$ is very small and positive, but the term inside the square root is large and positive. So one of the eigenvalues is positive, and the other is negative. Thus $(0, 0)$ is a saddle.

(d) Note that $dx/dt < 0$ on the positive y-axis and $dy/dt < 0$ on the positive x-axis. The signs are reversed on the negative axes. Hence, the eigenvectors for the negative eigenvalue are in the first and third quadrants and those for the poistive eigenvalue are in the second and fourth quadrants. Solutions starting near the origin in the first quadrant have either one or both species going extinct.



29. **(a)** At $(0,0)$, $\partial f/\partial x$ is positive and large, and $\partial g/\partial y$ is positive and small.

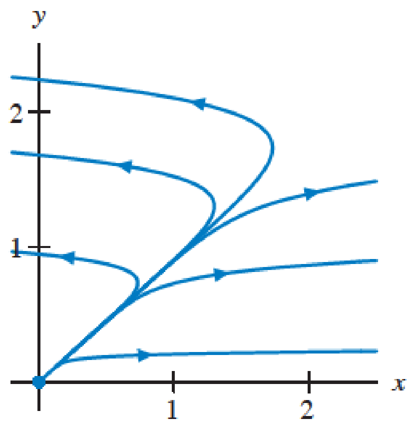
(b) At $(0,0)$, $\partial f/\partial y$ is negative with a large absolute value and $\partial g/\partial x = 0$.

(c) With these assumptions, the Jacobian matrix is

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

where $a > 0$, $b < 0$, and $d > 0$ is much smaller than a. The eigenvalues of this matrix are a and d, so $(0,0)$ is a source.

(d) Note that for $y = 0$, $dy/dt = 0$, and the eigenvector for a is in the x-direction.



30. **(a)** If z is fixed and y increases, then our assumption is that dy/dt decreases. That is, $\partial h/\partial y < 0$. Similarly, $\partial k/\partial z < 0$.

(b) Similarly, $\partial h/\partial z$ and $\partial k/\partial y$ are both positive.

(c) With these assumptions, the Jacobian matrix is

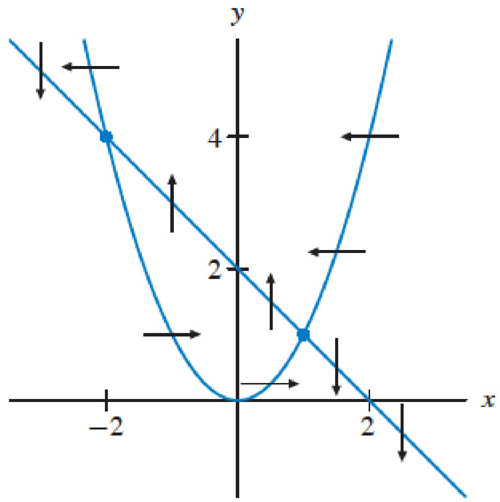
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a < 0$, $b > 0$, $c > 0$, and $d < 0$. The eigenvalues of this matrix are

$$\frac{a + d \pm \sqrt{(a-d)^2 + 4bc}}{2}.$$

These eigenvalues are always real, since the term inside the square root is positive. One eigenvalue is always negative (choose the negative square root). The other may be positive or negative. Thus, we only have saddles or sinks for equilibrium points.

1. For x- and y-nullclines, $dx/dt = 0$, and $dy/dt = 0$ respectively. Then, we obtain $y = -x + 2$ for the x-nullcline and $y = x^2$ for the y-nullcline. To find intersections, we set $-x + 2 = x^2$, or $(x+2)(x-1) = 0$. Solving this for x yields $x = 1, -2$. For $x = 1$, $y = 1$, and for $x = -2$, $y = 4$. So the equilibrium points are $(1, 1)$ and $(-2, 4)$.



The solution for (a) is in the left-down region, and therefore, it eventually enters the region where $y < -x + 2$ and $y < x^2$. Once the solution enters this region, it stays there because the vector field on the boundaries never points out. Solutions for (b) and (c) start in this same region. Hence, all three solutions will go down and to the right without bound.

4. **(a)** Equilibria are located where x-nullclines and y-nullclines intersect, so those equilibria with both $x > 0$ and $y > 0$ are located on the intersection of the lines

$$Ax + By = C \quad \text{and} \quad Dx + Ey = F.$$

However, the only way that two lines can intersect at more than one point is if they are really the same line. This happens if

$$A/D = B/E = C/F.$$

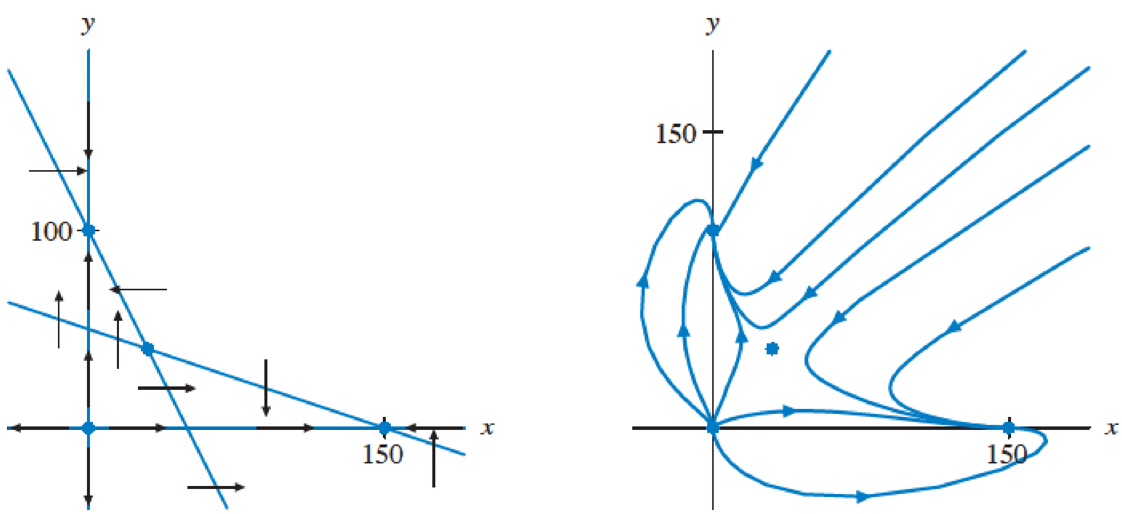
(b) To guarantee that there is exactly one equilibrium point at which the species coexist, we can stipulate that the x- and y-intercepts of the x- and y-nullclines are positioned so that these two lines are forced to intersect in the first quadrant. For example, we could require that the y-intercept of the x-nullcline, namely C/B, lies below the y-intercept of the y-nullcline, namely F/E, whereas the opposite happens for the x-intercepts. That is, we could require that

$$F/E > C/B \quad \text{but} \quad F/D < C/A.$$

Reversing both of these inequalities also guarantees that the species can coexist.

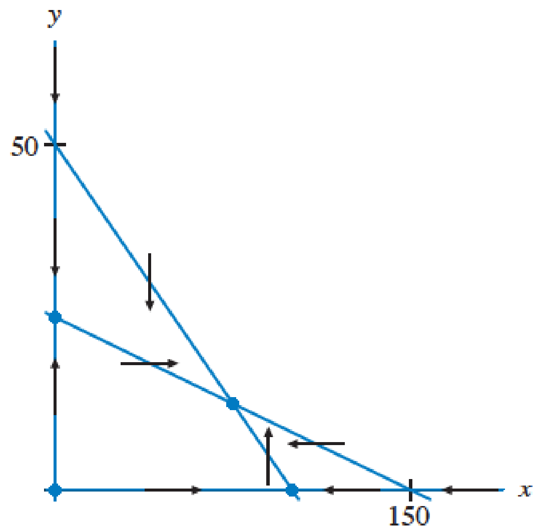
5. **(a)** The x-nullcline is made up of the lines $x = 0$ and $y = -x/3 + 50$. The y-nullcline is made up of the lines $y = 0$ and $y = -2x + 100$.

(b)



(c) Most solutions tend toward one of the equilibrium points (0, 100) or (150, 0). One curve of solutions divides these two behaviors. On this curve, solutions tend toward the saddle equilibrium at (30, 40).

7. **(a)** The x-nullcline consists of the two lines $x = 0$ and $y = -x/2 + 50$. The y-nullcline consists of the two lines $y = 0$ and $y = -x/6 + 25$.



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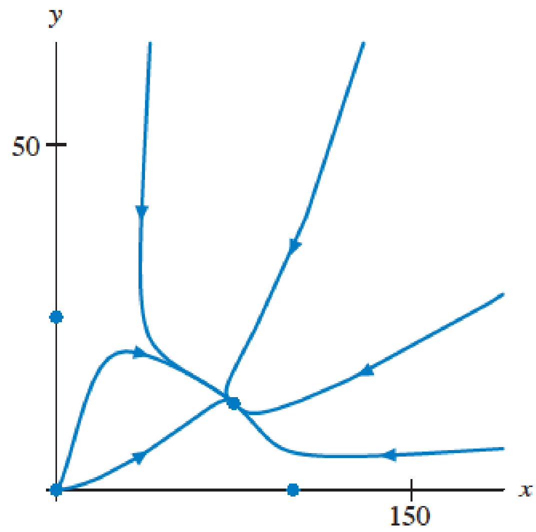
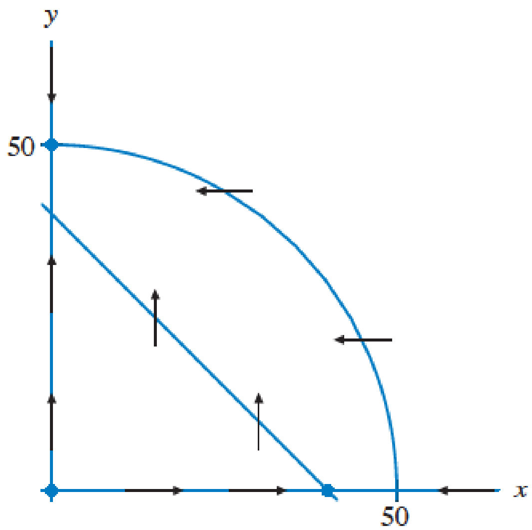


(c) All solutions off the axes tend toward the sink at $(75, 25/2)$. On the x-axis, solutions tend to the saddle at $(100, 0)$. On the y-axis, solutions tend to the saddle at $(0, 25)$.

9. **(a)** The x-nullcline is given by the two lines $x = 0$ and $y = -x + 40$. The y-nullcline is given by the line $y = 0$ and the circle $x^2 + y^2 = 50^2$.



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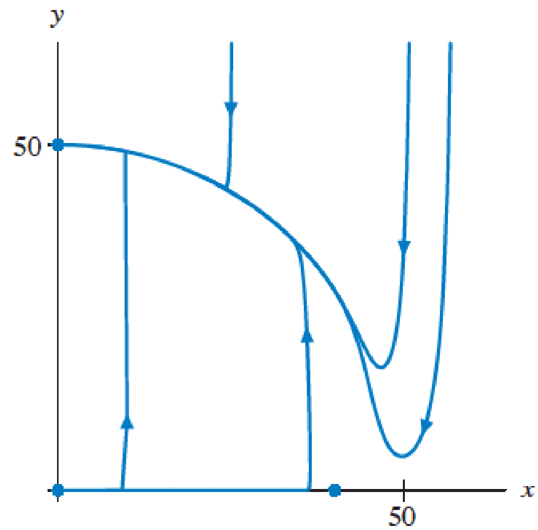
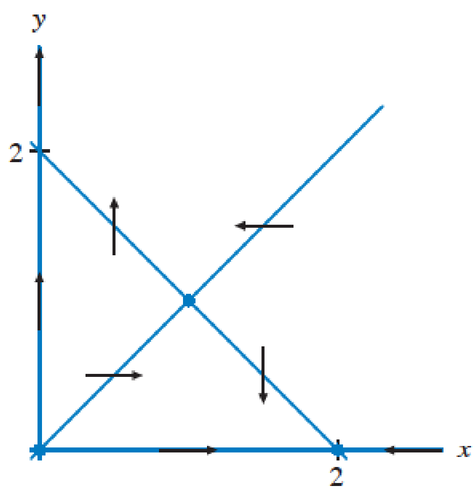
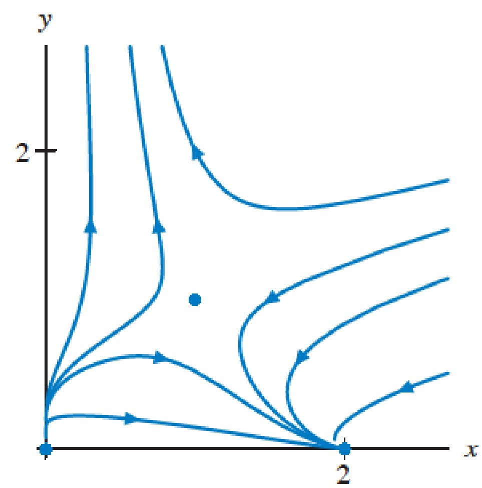


(c) Solutions off the x-axis tend toward the sink at $(0, 50)$. Solutions on the x-axis tend toward the saddle at $(40, 0)$.

13. **(a)** The x-nullcline is given by the lines $x = 0$ and $y = -x + 2$. The y-nullcline is given by the lines $y = 0$ and $y = x$.



(b)



(c) Most solutions tend toward either the sink at $(2, 0)$ or toward infinity in the y-direction (with $x < 1$). The curve separating these two behaviors is a curve of solutions that tend toward the saddle at $(1, 1)$.

15. **(a)** Since the species are cooperative, an increase in y results in an increase in x and vice versa. Therefore, one needs to change the signs in front of B and D from $-$ to $+$.

(b) The x-nullcline is given by $x = 0$ or $-Ax + By + C = 0$. The y-nullcline is given by $y = 0$ or $Dx - Ey + F = 0$. The origin is always an equilibrium point. Also, $x = 0$, $y = F/E$ and $x = C/A$, $y = 0$ are equilibrium points. Equilibrium points with both x and y positive arise from solutions of

$$\begin{cases} -Ax + By + C = 0 \\ Dx - Ey + F = 0 \end{cases}$$

In matrix notation, we obtain

$$\begin{pmatrix} -A & B \\ D & -E \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -C \\ -F \end{pmatrix}.$$

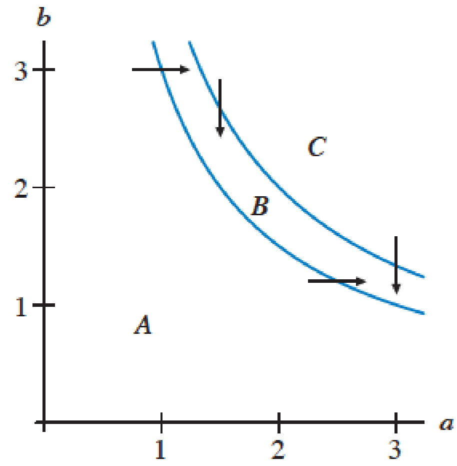
In order for a unique solution to exist, $AE - BD \neq 0$. Then, the solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{AE - BD} \begin{pmatrix} CE + BF \\ CD + AF \end{pmatrix}$$

Since A through F are all positive, we must have $AE - BD > 0$ for the solution to be in the first quadrant.

If $AE - BD = 0$, then $-Ax + By$ must be a negative multiple of $Dx - Ey$, so there are no solutions with both x and y positive.

17. **(a)** For the a-nullcline, $da/dt = 0$, so $2 - ab/2 = 0$, or $ab = 4$. For the b-nullcline, $db/dt = 0$, so $ab = 3$. Both nullclines are hyperbolas, and the curve of $ab = 4$ is above the one of $ab = 3$. Therefore, the direction of vector field on $ab = 4$ is vertical and downward, and the one on $ab = 3$ is horizontal and to the right.



(b) Below and above $ab = 3$, $db/dt > 0$ and $db/dt < 0$ respectively. Below and above $ab = 3$, $da/dt > 0$ and $da/dt < 0$ respectively. Therefore, in region A, the vector field points up and to the right, in region B, the vector field points down and to the right, and in region C, the vector field points down and to the left.

(c) On the boundaries of B, the direction of the vector field never points out of B. Therefore, as time increases, these solutions are asymptotic to the positive x-axis from above.