5. To show a rule by induction, we need two steps. First, we need to show the rule is true for $n = 1$. Then, we need to show that if the rule holds for n, then it holds for $n + 1$.

(a) $n = 1$. We need to show that $\mathcal{L}[t] = 1/s^2$.

$$\mathcal{L}[t] = \int_0^\infty te^{-st}\,dt.$$

Using integration by parts with $u = t$ and $dv = e^{-st}\,dt$, we find

$$\begin{aligned}
\mathcal{L}[t] &= \frac{te^{-st}}{-s}\bigg|_0^\infty + \int_0^\infty \frac{e^{-st}}{s}\,dt \\
&= \lim_{b\to\infty}\left[\frac{te^{-st}}{-s}\bigg|_0^b\right] + \int_0^\infty \frac{e^{-st}}{s}\,dt \\
&= \int_0^\infty \frac{e^{-st}}{s}\,dt \\
&= -\frac{e^{-st}}{s^2}\bigg|_0^\infty \\
&= \frac{1}{s^2} \quad (s > 0).
\end{aligned}$$

(b) Now we assume that the rule holds for n, that is, that $\mathcal{L}[t^n] = n!/s^{n+1}$, and show it holds true for $n+1$, that is, $\mathcal{L}[t^{n+1}] = (n+1)!/s^{n+2}$. There are two different methods to do so:

(i)

$$\mathcal{L}[t^{n+1}] = \int_0^\infty t^{n+1} e^{-st}\, dt$$

Using integration by parts with $u = t^{n+1}$ and $dv = e^{-st}\, dt$, we find

$$\mathcal{L}[t^{n+1}] = -\frac{t^{n+1}e^{-st}}{s}\bigg|_0^\infty + \int_0^\infty \frac{n+1}{s} t^n\, e^{-st}\, dt.$$

Now,

$$\begin{aligned}
-\frac{t^{n+1}e^{-st}}{s}\bigg|_0^\infty &= \lim_{b\to\infty}\left[-\frac{t^{n+1}e^{-st}}{s}\bigg|_0^b\right] \\
&= \lim_{b\to\infty}\frac{-b^{n+1}e^{-sb}}{s} + 0 \\
&= 0 \quad (s > 0).
\end{aligned}$$

So

$$\begin{aligned}
\mathcal{L}[t^{n+1}] &= \int_0^\infty \frac{n+1}{s} t^n\, e^{-st}\, dt \\
&= \frac{n+1}{s}\int_0^\infty t^n\, e^{-st}\, dt \\
&= \frac{n+1}{s}\mathcal{L}[t^n].
\end{aligned}$$

Since we assumed that $\mathcal{L}[t^n] = n!/s^{n+1}$, we get that

$$\mathcal{L}[t^{n+1}] = \frac{n+1}{s}\cdot\frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}$$

which is what we wanted to show.

(ii) We use the fact that $\mathcal{L}[df/dt] = s\mathcal{L}[f] - f(0)$. Letting $f(t) = t^{n+1}$ we have $f(0) = 0$ and

$$\mathcal{L}[(n+1)t^n] = s\mathcal{L}[t^{n+1}] - 0$$

or

$$(n+1)\mathcal{L}[t^n] = s\mathcal{L}[t^{n+1}]$$

using the fact that the Laplace transform is linear. Since we assumed $\mathcal{L}[t^n] = n!/s^{n+1}$, we have

$$\mathcal{L}[t^{n+1}] = \frac{n+1}{s}\mathcal{L}[t^n] = \frac{n+1}{s}\cdot\frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}},$$

which is what we wanted to show.

9. We see that

$$\frac{2}{3s+5} = \frac{2}{3} \cdot \frac{1}{s+5/3},$$

so

$$\mathcal{L}^{-1}\left[\frac{2}{3s+5}\right] = \frac{2}{3}e^{-\frac{5}{3}t}.$$

11. Using the method of partial fractions, we write

$$\frac{4}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}.$$

Putting the right-hand side over a common denominator gives $A(s+3) + B(s) = 4$, which can be written as $(A+B)s + 3A = 4$. Thus, $A+B = 0$, and $3A = 4$. This gives $A = 4/3$ and $B = -4/3$, so

$$\mathcal{L}^{-1}\left[\frac{4}{s(s+3)}\right] = \mathcal{L}^{-1}\left[\frac{4/3}{s} - \frac{4/3}{s+3}\right].$$

Hence,

$$\mathcal{L}^{-1}\left[\frac{4}{s(s+3)}\right] = \frac{4}{3} - \frac{4}{3}e^{-3t}.$$

13. Using the method of partial fractions, we have

$$\frac{2s+1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}.$$

Putting the right-hand side over a common denominator gives $A(s-2) + B(s-1) = 2s+1$, which can be written as $(A+B)s + (-2A-B) = 2s+1$. So, $A+B = 2$, and $-2A-B = 1$. Thus $A = -3$ and $B = 5$, which gives

$$\mathcal{L}^{-1}\left[\frac{2s+1}{(s-1)(s-2)}\right] = \mathcal{L}^{-1}\left[\frac{5}{s-2} - \frac{3}{s-1}\right].$$

Finally,

$$\mathcal{L}^{-1}\left[\frac{2s+1}{(s-1)(s-2)}\right] = 5e^{2t} - 3e^{t}.$$

15. **(a)** We have

$$\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$$

and

$$\mathcal{L}[-y + e^{-2t}] = \mathcal{L}[-y] + \mathcal{L}[e^{-2t}] = -\mathcal{L}[y] + \frac{1}{s+2}$$

using linearity of the Laplace transform and the formula $\mathcal{L}[e^{at}] = 1/(s-a)$ from the text.

(b) Substituting the initial condition yields

$$s\mathcal{L}[y] - 2 = -\mathcal{L}[y] + \frac{1}{s+2}$$

so that

$$(s+1)\mathcal{L}[y] = 2 + \frac{1}{s+2}$$

which gives

$$\mathcal{L}[y] = \frac{1}{(s+1)(s+2)} + \frac{2}{s+1} = \frac{2s+5}{(s+1)(s+2)}.$$

(c) Using the method of partial fractions,

$$\frac{2s+5}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}.$$

Putting the right-hand side over a common denominator gives $A(s+2) + B(s+1) = 2s+5$, which can be written as $(A+B)s + (2A+B) = 2s+5$. So we have $A+B = 2$, and $2A+B = 5$. Thus, $A = 3$ and $B = -1$, and

$$\mathcal{L}[y] = \frac{3}{s+1} - \frac{1}{s+2}.$$

Therefore, $y(t) = 3e^{-t} - e^{-2t}$ is the desired function.

(d) Since $y(0) = 3e^0 - e^0 = 2$, $y(t)$ satisfies the given initial condition. Also,

$$\frac{dy}{dt} = -3e^{-t} + 2e^{-2t}$$

and

$$-y + e^{-2t} = -3e^{-t} + e^{-2t} + e^{-2t} = -3e^{-t} + 2e^{-2t},$$

so our solution also satisfies the differential equation.

17. **(a)** Taking Laplace transforms of both sides of the equation and simplifying gives

$$\mathcal{L}\left[\frac{dy}{dt}\right] + 7\mathcal{L}[y] = \mathcal{L}[1]$$

so

$$s\mathcal{L}[y] - y(0) + 7\mathcal{L}[y] = \frac{1}{s}$$

and $y(0) = 3$ gives

$$s\mathcal{L}[y] - 3 + 7\mathcal{L}[y] = \frac{1}{s}.$$

(b) Solving for $\mathcal{L}[y]$ gives

$$\mathcal{L}[y] = \frac{3}{s+7} + \frac{1}{s(s+7)} = \frac{3s+1}{s(s+7)}.$$

(c) Using the method of partial fractions, we get

$$\frac{3s+1}{s(s+7)} = \frac{A}{s} + \frac{B}{s+7}.$$

Putting the right-hand side over a common denominator gives $A(s+7) + Bs = 3s + 1$, which can be written as $(A + B)s + 7A = 3s + 1$. So $A + B = 3$, and $7A = 1$. Hence, $A = 1/7$ and $B = 20/7$, and we have

$$\mathcal{L}[y] = \frac{1/7}{s} + \frac{20/7}{s+7}.$$

Thus,

$$y(t) = \frac{20}{7}e^{-7t} + \frac{1}{7}.$$

(d) To check, we compute

$$\frac{dy}{dt} + 7y = -20e^{-7t} + 7\left(\frac{20}{7}e^{-7t} + \frac{1}{7}\right) = 1,$$

and $y(0) = 20/7 + 1/7 = 3$, so our solution satisfies the initial-value problem.

23. **(a)** We have

$$\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$$

and

$$\mathcal{L}[-y+t^2] = \mathcal{L}[-y] + \mathcal{L}[t^2] = -\mathcal{L}[y] + \frac{2}{s^3}$$

using linearity of the Laplace transform and the formula $\mathcal{L}[t^n] = n!/s^{n+1}$ from Exercise 5.

(b) Substituting the initial condition yields

$$s\mathcal{L}[y] - 1 = -\mathcal{L}[y] + \frac{2}{s^3}$$

so that

$$\mathcal{L}[y] = \frac{2/s^3+1}{1+s} = \frac{2+s^3}{s^3(s+1)}.$$

(c) The best way to deal with this problem is with partial fractions. We seek constants A, B, C, and D such that

$$\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+1} = \frac{2+s^3}{s^3(s+1)}.$$

Multiplying through by $s^3(s+1)$ and equating like terms yields the system of equations

$$\begin{cases} A + D = 1 \\ A + B = 0 \\ B + C = 0 \\ C = 2. \end{cases}$$

Solving simultaneously gives us $A = 2$, $B = -2$, $C = 2$, and $D = -1$. Therefore we seek a function $y(t)$ whose Laplace transform is

$$\frac{2}{s} - \frac{2}{s^2} + \frac{2}{s^3} - \frac{1}{s+1}.$$

We have $\mathcal{L}[e^{-t}] = 1/(s+1)$ so that

$$\mathcal{L}[-e^{-t}] = -\mathcal{L}[e^{-t}] = -\frac{1}{s+1}.$$

Also, using the formula from Exercise 5, we have

$$\mathcal{L}[t^2] = \frac{2}{s^3}, \quad \mathcal{L}[t] = \frac{1}{s^2}, \quad \text{and} \quad \mathcal{L}[1] = \frac{1}{s}$$

so that

$$\mathcal{L}[t^2 - 2t + 2] = \mathcal{L}[t^2] - 2\mathcal{L}[t] + 2\mathcal{L}[1] = \frac{2}{s^3} - \frac{2}{s^2} + \frac{2}{s}.$$

Therefore, $y(t) = t^2 - 2t + 2 - e^{-t}$ is the desired function.

(d) Since $y(0) = 2 - e^0 = 1$, $y(t)$ satisfies the given initial condition. Also,

$$\frac{dy}{dt} = 2t - 2 + e^{-t}$$

and

$$-y + t^2 = -t^2 + 2t - 2 + e^{-t} + t^2 = 2t - 2 + e^{-t}$$

so our solution also satisfies the differential equation.

1. **(a)** The function $g_a(t) = 1$ precisely when $u_a(t) = 0$, and $g_a(t) = 0$ precisely when $u_a(t) = 1$, so

$$g_a(t) = 1 - u_a(t).$$

(b) We can compute the Laplace transform of $g_a(t)$ from the definition

$$\mathcal{L}[g_a] = \int_0^a 1e^{-st}\,dt = -\frac{e^{-as}}{s} + \frac{e^{-0s}}{s} = \frac{1}{s} - \frac{e^{-as}}{s}.$$

Alternately, we can use the table

$$\mathcal{L}[g_a] = \mathcal{L}[1 - u_a(t)] = \frac{1}{s} - \frac{e^{-as}}{s}.$$

5. First use partial fractions to write

$$\frac{1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}.$$

Putting the right-hand side over a common denominator yields $As - 2A + Bs - B = 1$ which can be written as $(A + B)s + (-2A - B) = 1$. Thus, $A + B = 0$, and $-2A - B = 1$. Solving for A and B yields $A = -1$ and $B = 1$, so

$$\frac{1}{(s-1)(s-2)} = \frac{-1}{s-1} + \frac{1}{s-2}.$$

Now, as above

$$\mathcal{L}[u_3(t)e^{2(t-3)}] = \frac{e^{-3s}}{s-2}$$

and

$$\mathcal{L}[u_3(t)e^{t-3}] = \frac{e^{-3s}}{s-1}$$

and the desired function is

$$u_3(t)\left(e^{2(t-3)} - e^{(t-3)}\right).$$

9. Taking the Laplace transform of both sides of the equation, we have

$$\mathcal{L}\left[\frac{dy}{dt}\right] + 9\mathcal{L}[y] = \mathcal{L}[u_5(t)],$$

which is equivalent to

$$s\mathcal{L}[y] - y(0) + 9\mathcal{L}[y] = \frac{e^{-5s}}{s}.$$

Since $y(0) = -2$, we have

$$s\mathcal{L}[y] + 2 + 9\mathcal{L}[y] = \frac{e^{-5s}}{s},$$

which yields

$$\mathcal{L}[y] = \frac{-2}{s+9} + \frac{e^{-5s}}{s(s+9)}.$$

Using the partial fractions decomposition

$$\frac{1}{s(s+9)} = \frac{1/9}{s} - \frac{1/9}{s+9},$$

we see that

$$\mathcal{L}[y] = \frac{-2}{s+9} + \frac{1}{9}\left(\frac{e^{-5s}}{s}\right) - \frac{1}{9}\left(\frac{e^{-5s}}{s+9}\right).$$

Taking the inverse of the Laplace transform, we obtain

$$\begin{aligned} y(t) &= -2e^{-9t} + \frac{1}{9}u_5(t) - \frac{1}{9}u_5(t)e^{-9(t-5)} \\ &= -2e^{-9t} + \frac{1}{9}u_5(t)\left(1 - e^{-9(t-5)}\right). \end{aligned}$$

To check our answer, we compute

$$\frac{dy}{dt} = 18e^{-9t} + \frac{1}{9}\frac{du_5}{dt}\left(1 - e^{-9(t-5)}\right) + \frac{1}{9}u_5(t)\left(9e^{-9(t-5)}\right),$$

and since $du_5/dt = 0$ except at $t = 5$ (where it is undefined),

$$\begin{aligned} \frac{dy}{dt} + 9y &= 18e^{-9t} + u_5(t)e^{-9(t-5)} + 9\left(-2e^{-9t} + \frac{1}{9}u_5(t)\left(1 - e^{-9(t-5)}\right)\right) \\ &= u_5(t). \end{aligned}$$

Hence, our $y(t)$ satisfies the differential equation except when $t = 5$. (We cannot expect $y(t)$ to satisfy the differential equation at $t = 5$ because the differential equation is not continuous there.) Note that $y(t)$ also satisfies the initial condition $y(0) = -2$.