1. (21 points) Short answer questions: The answers to these questions need only consist of one or two sentences. Partial credit will be awarded only in exceptional situations.

(a) Consider the harmonic oscillator with mass 2, spring constant 5, and damping coefficient b. Find the values of b for which the system is overdamped, critically damped, underdamped, and undamped.

$b = 0$ undamped

underdamped $b^2 - 40 < 0 \Rightarrow 0 < b < \sqrt{40}$

critically damped $b = \sqrt{40}$

overdamped $b > \sqrt{40}$

(b) Find one solution of the forced harmonic oscillator

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 1.$$

The constant function $y(t) = \frac{1}{k}$ for all t satisfies the equation because $\frac{dy}{dt} = 0$ and $\frac{d^2y}{dt^2} = 0$.

(c) For the nonlinear system $dx/dt = x^2 + \sin 3x$ and $dy/dt = 2y - \sin xy$, determine the linearized system at the origin and determine the type (sink, source, ...) of the equilibrium point there.

linearized system: $\frac{dx}{dt} = 3x$

$\frac{dy}{dt} = 2y$

Eigenvalues are 2 and 3 $\Rightarrow$ the origin is a source.

2. (18 points) Consider the second-order equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x + x^3 = 0.$$

(a) Convert this differential equation to a first-order system and calculate its equilibrium point(s).

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = -2v + 3x - x^3$$

Eq point $\Rightarrow v = 0 \Rightarrow 3x - x^3 = 0$

$x(3 - x^2) = 0$

Three equilibrium points:

$(x,v) = (0,0) \quad (x,v) = (\sqrt{3}, 0) \quad (x,v) = (-\sqrt{3}, 0)$

(b) Determine the types of these equilibria using linearization.

Jacobian $J(x,v) = \begin{pmatrix} 0 & 1 \\ 3-3x^2 & -2 \end{pmatrix}$

$J(0,0) = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix}$ $\det J = -3 \Rightarrow$ saddle

$J(\pm\sqrt{3}, 0) = \begin{pmatrix} 0 & 1 \\ -6 & -2 \end{pmatrix}$ $\lambda^2 + 2\lambda + 6 = 0$

$\lambda = \dfrac{-2 \pm \sqrt{4 - 24}}{2}$

both complex with neg. real parts $\Rightarrow$ spiral sink.

3. (20 points) Consider the second-order equation

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = 6t + 3e^{4t}.$$

(a) Determine a particular solution to this differential equation.

Guess $y_p(t) = a + bt + ce^{4t}$

$$\frac{dy_p}{dt} = b + 4ce^{4t}$$

$$\frac{d^2y_p}{dt^2} = 16ce^{4t}$$

$$\frac{d^2y_p}{dt^2} - \frac{dy_p}{dt} - 6y_p = 16ce^{4t} - b - 4ce^{4t} - 6a - 6bt - 6ce^{4t}$$

$$= 6ce^{4t} - 6bt - 6a - b$$

$$\stackrel{?}{=} 6t + 3e^{4t}$$

$$\Rightarrow c = \frac{1}{2}, \quad b = -1, \quad a = -\frac{b}{6} = \frac{1}{6}$$

(b) Find the general solution to this differential equation.

$$\lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) \Rightarrow \lambda = 3, -2$$

$$y(t) = k_1 e^{3t} + k_2 e^{-2t} + \frac{1}{6} - t + \frac{1}{2} e^{4t}$$

4. (18 points) Solve the initial-value problem

$$\frac{dy}{dt} - 3y = u_2(t)\,e^{-4(t-2)}, \quad y(0) = 1.$$

$$\mathcal{L}\left[\frac{dy}{dt} - 3y\right] = \mathcal{L}\left[u_2(t)\,e^{-4(t-2)}\right]$$

$$\mathcal{L}\left[\frac{dy}{dt}\right] - 3\mathcal{L}[y] = e^{-2s}\left(\frac{1}{s+4}\right)$$

$$s\mathcal{L}[y] - 1 - 3\mathcal{L}[y] = \frac{e^{-2s}}{s+4}$$

$$(s-3)\mathcal{L}[y] = 1 + \frac{e^{-2s}}{s+4}$$

$$\mathcal{L}[y] = \frac{1}{s-3} + \frac{e^{-2s}}{(s-3)(s+4)}$$

$$\frac{1}{(s-3)(s+4)} = \frac{A}{s-3} + \frac{B}{s+4} \implies \begin{array}{l} A+B=0 \\ 4A-3B=1 \\ 7A=1 \\ A=\frac{1}{7},\ B=-\frac{1}{7} \end{array}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s-3)(s+4)}\right] = \frac{1}{7}e^{3t} - \frac{1}{7}e^{-4t}$$

$$= \frac{1}{7}\left(e^{3t} - e^{-4t}\right)$$

$$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{(s-3)(s+4)}\right] = \frac{u_2(t)}{7}\left(e^{3(t-2)} - e^{-4(t-2)}\right)$$

$$\implies y(t) = e^{3t} + \frac{u_2(t)}{7}\left(e^{3(t-2)} - e^{-4(t-2)}\right)$$

5. (20 points) Consider the one-parameter family of equations

$$m\frac{d^2y}{dt^2} + \frac{dy}{dt} + 2y = 0,$$

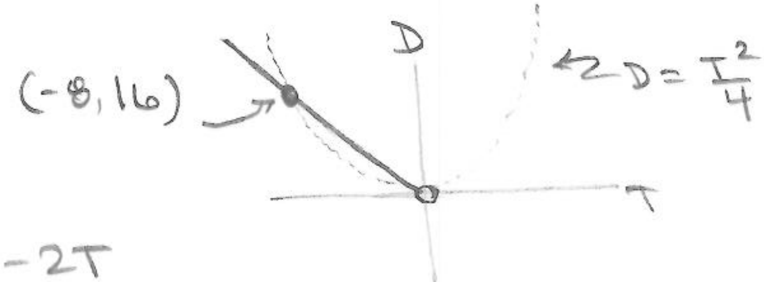
where m is a **positive** parameter.

(a) Convert the family into a one-parameter family of first-order systems.

$$\frac{dy}{dt} = v \qquad \frac{dv}{dt} = \frac{d^2y}{dt^2} = -\frac{2}{m}y - \frac{1}{m}v$$

(b) Draw the curve in the trace-determinant plane obtained by varying the parameter.

$$\begin{pmatrix} 0 & 1 \\ -\frac{2}{m} & -\frac{1}{m} \end{pmatrix}$$



trace $T = -\frac{1}{m}$

det $D = \frac{2}{m}$ $\Rightarrow D = -2T$

$\frac{T^2}{4} = -2T \Rightarrow T^2 + 8T = 0 \Rightarrow T = 0, -8$

(c) Determine all bifurcation values and briefly discuss the different types of behavior exhibited by this one-parameter family.

Can use part (b) to answer this part, but we use the characteristic polynomial

$$m\lambda^2 + \lambda + 2 = 0$$

$$\Rightarrow \lambda = \frac{-1 \pm \sqrt{1-8m}}{2m}$$

$0 < m < \frac{1}{8} \Rightarrow$ real eigenvalues (overdamped)

$m = \frac{1}{8}$ is a bifurcation value (critically damped)

$m > \frac{1}{8} \Rightarrow$ complex eigenvalues (underdamped)