EXERCISES FOR SECTION 3.1

1. Since $a > 0$, Paul's making a profit ($x > 0$) has a beneficial effect on Paul's profits in the future because the ax term makes a positive contribution to dx/dt. However, since $b < 0$, Bob's making a profit ($y > 0$) hinders Paul's ability to make profit because the by term contributes negatively to dx/dt. Roughly speaking, business is good for Paul if his store is profitable and Bob's is not. In fact, since $dx/dt = x - y$, Paul's profits will increase whenever his store is more profitable than Bob's.

Even though $dx/dt = dy/dt = x - y$ for this choice of parameters, the interpretation of the equation is exactly the opposite from Bob's point of view. Since $d < 0$, Bob's future profits are hurt whenever he is profitable because $dy < 0$. But Bob's profits are helped whenever Paul is profitable since $cx > 0$. Once again, since $dy/dt = x - y$, Bob's profits will increase whenever Paul's store is more profitable than his.

Finally, note that both x and y change by identical amounts since dx/dt and dy/dt are always equal.

2. Since $a = 2$, Paul's making a profit ($x > 0$) has a beneficial effect on Paul's future profits because the ax term makes a positive contribution to dx/dt. However, since $b = -1$, Bob's making a profit ($y > 0$) hinders Paul's ability to make profit because the by term contributes negatively to dx/dt. In some sense, Paul's profitability has twice the impact on his profits as does Bob's profitability. For example, Paul's profits will increase whenever his profits are at least one-half of Bob's profits since $dx/dt = 2x - y$.

Since $c = d = 0$, $dy/dt = 0$. Consequently, Bob's profits are not affected by the profitability of either store, and hence his profits are constant in this model.

3. Since $a = 1$ and $b = 0$, we have $dx/dt = x$. Hence, if Paul is making a profit ($x > 0$), then those profits will increase since dx/dt is positive. However, Bob's profits have no effect on Paul's profits. (Note that $dx/dt = x$ is the standard exponential growth model.)

Since $c = 2$ and $d = 1$, profits from both stores have a positive effect on Bob's profits. In some sense, Paul's profits have twice the impact of Bob's profits on dy/dt.

4. Since $a = -1$ and $b = 2$, Paul's making a profit has a negative effect on his future profits. However, if Bob makes a profit, then Paul's profits benefit. Moreover, Bob's profitability has twice the impact as does Paul's. In fact, since $dx/dt = -x + 2y$, Paul's profits will increase if $-x + 2y > 0$ or, in other words, if Bob's profits are at least one-half of Paul's profits.

Since $c = 2$ and $d = -1$, Bob is in the same situation as Paul. His profits contribute negatively to dy/dt since $d = -1$. However, Paul's profitability has twice the positive effect.

Note that this model is symmetric in the sense that both Paul and Bob perceive each others profits in the same way. This symmetry comes from the fact that $a = d$ and $b = c$.

5. $\mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\mathbf{Y}$

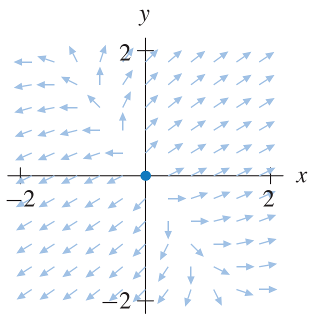
6. $\mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 3 \\ -0.3 & 3\pi \end{pmatrix}\mathbf{Y}$

7. $\mathbf{Y} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \quad \dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 3 & -2 & -7 \\ -2 & 0 & 6 \\ 0 & 7.3 & 2 \end{pmatrix}\mathbf{Y}$

8.
$$\frac{dx}{dt} = -3x + 2\pi y$$
$$\frac{dy}{dt} = 4x - y$$

9.
$$\frac{dx}{dt} = \beta y$$
$$\frac{dy}{dt} = \gamma x - y$$

10. **(a)**



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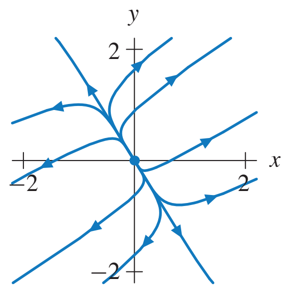

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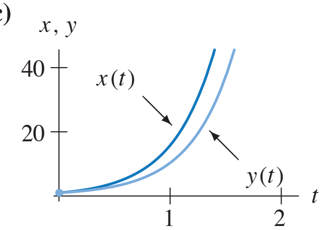
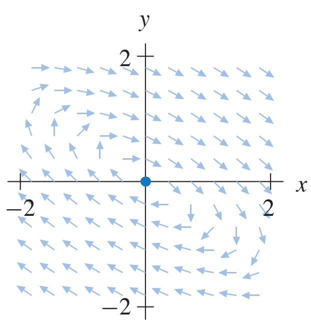


11. **(a)**



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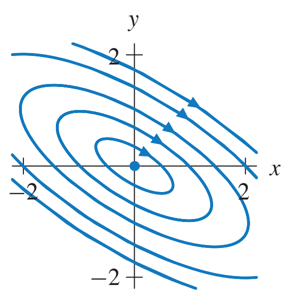

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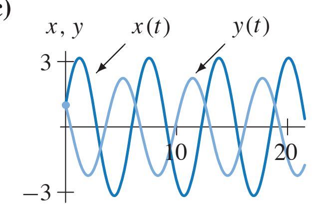
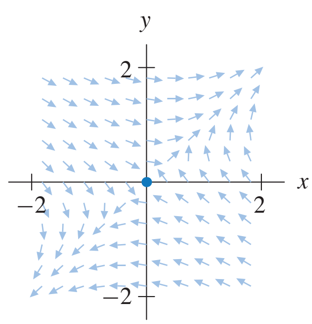


12. **(a)**



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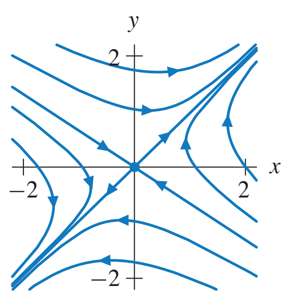
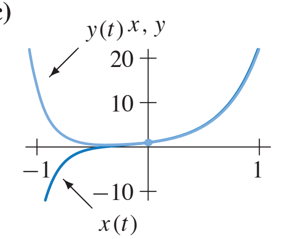
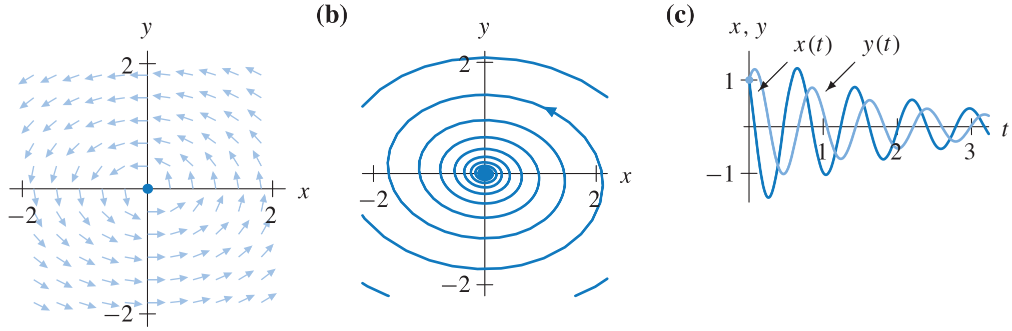


(c)



13. **(a)** **(b)** **(c)**



14. **(a)** If $a = 0$, then $\det \mathbf{A} = ad - bc = bc$. Thus both b and c are nonzero if $\det \mathbf{A} \neq 0$.

(b) Equilibrium points (x_0, y_0) are solutions of the simultaneous system of linear equations

$$\begin{cases} ax_0 + by_0 = 0 \\ cx_0 + dy_0 = 0. \end{cases}$$

If $a = 0$, the first equation reduces to $by_0 = 0$, and since $b \neq 0$, $y_0 = 0$. In this case, the second equation reduces to $cx_0 = 0$, so $x_0 = 0$ as well. Therefore, $(x_0, y_0) = (0, 0)$ is the only equilibrium point for the system.

15. The vector field at a point (x_0, y_0) is (ax_0+by_0, cx_0+dy_0), so in order for a point to be an equilibrium point, it must be a solution to the system of simultaneous linear equations

$$\begin{cases} ax_0 + by_0 = 0 \\ cx_0 + dy_0 = 0. \end{cases}$$

If $a \neq 0$, we know that the first equation is satisfied if and only if

$$x_0 = -\frac{b}{a} y_0.$$

Now we see that any point that lies on this line $x_0 = (-b/a)y_0$ also satisfies the second linear equation $cx_0 + dy_0 = 0$. In fact, if we substitute a point of this form into the second component of the vector field, we have

$$\begin{aligned} cx_0 + dy_0 &= c\left(-\frac{b}{a}\right) y_0 + dy_0 \\ &= \left(-\frac{bc}{a} + d\right) y_0 \\ &= \left(\frac{ad - bc}{a}\right) y_0 \\ &= \frac{\det \mathbf{A}}{a} y_0 \\ &= 0, \end{aligned}$$

since we are assuming that $\det \mathbf{A} = 0$. Hence, the line $x_0 = (-b/a)y_0$ consists entirely of equilibrium points.

If $a = 0$ and $b \neq 0$, then the determinant condition $\det \mathbf{A} = ad - bc = 0$ implies that $c = 0$. Consequently, the vector field at the point (x_0, y_0) is (by_0, dy_0). Since $b \neq 0$, we see that we get equilibrium points if and only if $y_0 = 0$. In other words, the set of equilibrium points is exactly the x-axis.

Finally, if $a = b = 0$, then the vector field at the point (x_0, y_0) is $(0, cx_0 + dy_0)$. In this case, we see that a point (x_0, y_0) is an equilibrium point if and only if $cx_0 + dy_0 = 0$. Since at least one of c or d is nonzero, the set of points (x_0, y_0) that satisfy $cx_0 + dy_0 = 0$ is precisely a line through the origin.

16. **(a)** Let $v = dy/dt$. Then $dv/dt = d^2y/dt^2 = -qy - p(dy/dt) = -qy - pv$. Thus we obtain the system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -qy - pv.$$

In matrix form, this system is written as

$$\begin{pmatrix} \frac{dy}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix}.$$

(b) The determinant of this matrix is q. Hence, if $q \neq 0$, we know that the only equilibrium point is the origin.

(c) If y is constant, then $v = dy/dt$ is identically zero. Hence, $dv/dt = 0$.

Also, the system reduces to

$$\begin{pmatrix} \frac{dy}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} y \\ 0 \end{pmatrix},$$

which implies that $dv/dt = -qy$.

Combining these two observations, we obtain $dv/dt = -qy = 0$, and if $q \neq 0$, then $y = 0$.

17. The first-order system corresponding to this equation is

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -qy - pv.$$

(a) If $q = 0$, then the system becomes

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -pv,$$

and the equilibrium points are the solutions of the system of equations

$$\begin{cases} v = 0 \\ -pv = 0. \end{cases}$$

Thus, the point (y, v) is an equilibrium point if and only if $v = 0$. In other words, the set of all equilibria agrees with the horizontal axis in the yv-plane.

(b) If $p = q = 0$, then the system becomes

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = 0$$

but the equilibrium points are again the points with $v = 0$.

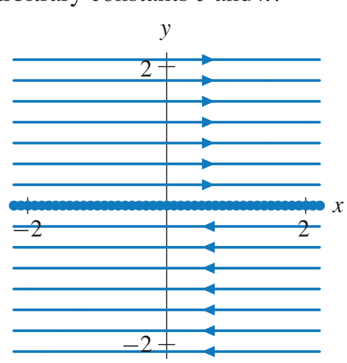
18. In this case, $dv/dt = d^2y/dt^2 = 0$, and the first-order system reduces to

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = 0.$$

(a) Since $dv/dt = 0$, we know that $v(t) = c$ for some constant c.

(b) Since $dy/dt = v = c$, we can integrate to obtain $y(t) = ct + k$ where k is another arbitrary constant. Hence, the general solution of the system consists of all functions of the form $(y(t), v(t)) = (ct + k, c)$ for arbitrary constants c and k.

(c)



19. Letting $v = dy/dt$ and $w = d^2y/dt^2$ we can write this equation as the system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = \frac{d^2y}{dt^2} = w$$
$$\frac{dw}{dt} = \frac{d^3y}{dt^3} = -ry - qv - pw.$$

In matrix notation, this system is

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

where

$$\mathbf{Y} = \begin{pmatrix} y \\ v \\ w \end{pmatrix} \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -r & -q & -p \end{pmatrix}.$$

20. If there are more than the usual number of buyers, then $b > 0$. If this level of buying means that prices will increase and that fewer buyers will enter the market, then the effect on db/dt should be negative. Since $db/dt = \alpha b + \beta s$, we expect that the αb-term will be negative if $b > 0$. Consequently, α should be negative.

21. If there are fewer than the usual number of buyers, then $b < 0$. If this level of b has a negative effect on the number of sellers, we expect the γb-term in ds/dt to be negative. If $\gamma b < 0$ and $b < 0$, then we must have $\gamma > 0$.

22. If $s > 0$, there are more than the usual number of houses for sale and house prices should decline. Declining prices should have a positive effect on the number of buyers and a negative effect on the number of sellers. Since $db/dt = \alpha b + \beta s$, we expect the βs-term to be positive. Since $\beta s > 0$ if $s > 0$, the parameter β should be positive.

23. In the model, $ds/dt = \gamma b + \delta s$. If $s > 0$, then the number of sellers is greater than usual and house prices should decline. Since declining prices should have a negative effect on the number of sellers, we expect the δs-term to be negative. If $\delta s < 0$ when $s > 0$, we should have $\delta < 0$.

24. **(a)** Substituting $\mathbf{Y}_1(t)$ in the left-hand side of the differential equation yields

$$\frac{d\mathbf{Y}_1}{dt} = \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

Moreover, the right-hand side becomes

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \mathbf{Y}_1(t) = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ e^t \end{pmatrix} = \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

Since the two sides of the differential equation agree, $\mathbf{Y}_1(t)$ is a solution.

Similarly, if we substitute $\mathbf{Y}_2(t)$ in the left-hand side of the differential equation, we get

$$\frac{d\mathbf{Y}_2}{dt} = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix}.$$

Moreover, the right-hand side is

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \mathbf{Y}_2(t) = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{2t} \\ e^{2t} + e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix}.$$

Since the two sides of the differential equation also agree for this function, $\mathbf{Y}_2(t)$ is another solution.

(b) At $t = 0$, $\mathbf{Y}(0) = (-2, -1)$. By the Linearity Principle, any linear combination of two solutions is also a solution. Hence, we solve the given initial-value problem with a function of the form $k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t)$ where k_1 and k_2 are constants determined by the initial value. That is, we determine k_1 and k_2 via

$$k_1\mathbf{Y}_1(0) + k_2\mathbf{Y}_2(0) = \mathbf{Y}(0) = \begin{pmatrix} -2 \\ -1 \end{pmatrix}.$$

We get

$$k_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}.$$

This vector equation is equivalent to the simultaneous linear equations

$$\begin{cases} \quad\quad k_2 = -2 \\ k_1 + k_2 = -1. \end{cases}$$

From the first equation, we have $k_2 = -2$. Then from the second equation, we obtain $k_1 = 1$. Therefore, the solution to the initial-value problem is

$$\begin{aligned} \mathbf{Y}(t) &= \mathbf{Y}_1(t) - 2\mathbf{Y}_2(t) \\ &= \begin{pmatrix} 0 \\ e^t \end{pmatrix} - 2 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} -2e^{2t} \\ e^t - 2e^{2t} \end{pmatrix}. \end{aligned}$$

Note that (as always) we can check our calculations directly. By direct evaluation, we know that $\mathbf{Y}(0) = (-2, -1)$. Moreover, we can check that $\mathbf{Y}(t)$ satisfies the differential equation. The left-hand side of the differential equation is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -4e^{2t} \\ e^t - 4e^{2t} \end{pmatrix},$$

and the right-hand side of the differential equation is

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \mathbf{Y}(t) = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -2e^{2t} \\ e^t - 4e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} -4e^{2t} \\ e^{t} - 4e^{2t} \end{pmatrix}.$$

Since the left-hand side and the right-hand side agree, the function $\mathbf{Y}(t)$ is a solution to the differential equation, and since it assumes the given initial value, this function is the desired solution to the initial-value problem. The Uniqueness Theorem says that this function is the only solution to the initial-value problem.

25. **(a)** Note that substituting $\mathbf{Y}(t)$ into the left-hand side of the differential equation, we get

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} e^{2t} + 2te^{2t} \\ -e^{2t} - 2(t+1)e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} e^{2t} + 2te^{2t} \\ -3e^{2t} - 2te^{2t} \end{pmatrix}.$$

Substituting $\mathbf{Y}(t)$ into the right-hand side, we get

$$\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} te^{2t} \\ -(t+1)e^{2t} \end{pmatrix} = \begin{pmatrix} te^{2t} + (t+1)e^{2t} \\ te^{2t} - 3(t+1)e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} e^{2t} + 2te^{2t} \\ -3e^{2t} - 2te^{2t} \end{pmatrix}.$$

Since the left-hand side of the differential equation equals the right-hand side, the function $\mathbf{Y}(t)$ is a solution.

(b) At $t = 0$, $\mathbf{Y}(0) = (0, -1)$. By the Linearity Principle, any constant multiple of the solution $\mathbf{Y}(t)$ is also a solution. Since the function $-2\mathbf{Y}(t)$ has the desired initial condition, we know that

$$-2\mathbf{Y}(t) = \begin{pmatrix} -2te^{2t} \\ (2t+2)e^{2t} \end{pmatrix}$$

is the desired solution. By the Uniqueness Theorem, this is the only solution with this initial condition. (Given the formula for $-2\mathbf{Y}(t)$ directly above, note that we can directly check our assertion that this function solves the initial-value problem without appealing to the Linearity Principle.)

26. **(a)** Substitute $\mathbf{Y}_1(t)$ into the differential equation and compare the left-hand side to the right-hand side. On the left-hand side, we have

$$\frac{d\mathbf{Y}_1}{dt} = \begin{pmatrix} -3e^{-3t} \\ -3e^{-3t} \end{pmatrix},$$

and on the right-hand side, we have

$$\mathbf{A}\mathbf{Y}_1(t) = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix} = \begin{pmatrix} -2e^{-3t} - e^{-3t} \\ 2e^{-3t} - 5e^{-3t} \end{pmatrix} = \begin{pmatrix} -3e^{-3t} \\ -3e^{-3t} \end{pmatrix}.$$

Since the two sides agree, we know that $\mathbf{Y}_1(t)$ is a solution.

For $\mathbf{Y}_2(t)$,

$$\frac{d\mathbf{Y}_2}{dt} = \begin{pmatrix} -4e^{-4t} \\ -8e^{-4t} \end{pmatrix},$$

and

$$\mathbf{A}\mathbf{Y}_2(t) = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} e^{-4t} \\ 2e^{-4t} \end{pmatrix} = \begin{pmatrix} -2e^{-4t} - 2e^{-4t} \\ 2e^{-4t} - 10e^{-4t} \end{pmatrix} = \begin{pmatrix} -4e^{-4t} \\ -8e^{-4t} \end{pmatrix}.$$

Since the two sides agree, the function $\mathbf{Y}_2(t)$ is also a solution.

Both $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions, and we proceed to the next part of the exercise.

(b) Note that $\mathbf{Y}_1(0) = (1, 1)$ and $\mathbf{Y}_2(0) = (1, 2)$. These vectors are not on the same line through the origin, so the initial conditions are linearly independent. If the initial conditions are linearly independent, then the solutions must also be linearly independent. Since the two solutions are linearly independent, we proceed to part (c) of the exercise.

(c) We must find constants k_1 and k_2 such that

$$k_1\mathbf{Y}_1(0) + k_2\mathbf{Y}_2(0) = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

In other words, the constants k_1 and k_2 must satisfy the simultaneous system of linear equations

$$\begin{cases} k_1 + k_2 = 2 \\ k_1 + 2k_2 = 3. \end{cases}$$

It follows that $k_1 = 1$ and $k_2 = 1$. Hence, the required solution is

$$\mathbf{Y}_1(t) + \mathbf{Y}_2(t) = \begin{pmatrix} e^{-3t} + e^{-4t} \\ e^{-3t} + 2e^{-4t} \end{pmatrix}.$$

27. **(a)** Substitute $\mathbf{Y}_1(t)$ into the differential equation and compare the left-hand side to the right-hand side. On the left-hand side, we have

$$\frac{d\mathbf{Y}_1}{dt} = \begin{pmatrix} -3e^{-3t} + 8e^{-4t} \\ -3e^{-3t} + 16e^{-4t} \end{pmatrix},$$

and on the right-hand side, we have

$$\mathbf{A}\mathbf{Y}_1(t) = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} e^{-3t} - 2e^{-4t} \\ e^{-3t} - 4e^{-4t} \end{pmatrix} = \begin{pmatrix} -3e^{-3t} + 8e^{-4t} \\ -3e^{-3t} + 16e^{-4t} \end{pmatrix}.$$

Since the two sides agree, we know that $\mathbf{Y}_1(t)$ is a solution.

For $\mathbf{Y}_2(t)$,

$$\frac{d\mathbf{Y}_2}{dt} = \begin{pmatrix} -6e^{-3t} - 4e^{-4t} \\ -6e^{-3t} - 8e^{-4t} \end{pmatrix},$$

and

$$\mathbf{A}\mathbf{Y}_2(t) = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} 2e^{-3t} + e{-}4t \\ 2e^{-3t} + 2e^{-4t} \end{pmatrix} = \begin{pmatrix} -6e^{-3t} - 4e^{-4t} \\ -6e^{-3t} - 8e^{-4t} \end{pmatrix}.$$

Since the two sides agree, the function $\mathbf{Y}_2(t)$ is also a solution.

Both $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions, and we proceed to the next part of the exercise.

(b) Note that $\mathbf{Y}_1(0) = (-1, -3)$ and $\mathbf{Y}_2(0) = (3, 4)$. These vectors are not on the same line through the origin, so the initial conditions are linearly independent. If the initial conditions are linearly independent, then the solutions must also be linearly independent. Since the two solutions are linearly independent, we proceed to part (c) of the exercise.

(c) We must find constants k_1 and k_2 such that

$$k_1\mathbf{Y}_1(0) + k_2\mathbf{Y}_2(0) = k_1 \begin{pmatrix} -1 \\ -3 \end{pmatrix} + k_2 \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

In other words, the constants k_1 and k_2 must satisfy the simultaneous system of linear equations

$$\begin{cases} -k_1 + 3k_2 = 2 \\ -3k_1 + 4k_2 = 3. \end{cases}$$

It follows that $k_1 = -1/5$ and $k_2 = 3/5$. Hence, the required solution is

$$-\frac{1}{5}\mathbf{Y}_1(t) + \frac{3}{5}\mathbf{Y}_2(t) = \begin{pmatrix} e^{-3t} + e^{-4t} \\ e^{-3t} + 2e^{-4t} \end{pmatrix}.$$

28. **(a)** First we substitute $\mathbf{Y}_1(t)$ into the differential equation. The left-hand side becomes

$$\begin{aligned} \frac{d\mathbf{Y}_1}{dt} &= -2e^{-2t} \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} + e^{-2t} \begin{pmatrix} -3\sin 3t \\ 3\cos 3t \end{pmatrix} \\ &= e^{-2t} \begin{pmatrix} -2\cos 3t - 3\sin 3t \\ 3\cos 3t - 2\sin 3t \end{pmatrix}, \end{aligned}$$

and the right-hand side is

$$\mathbf{A}\mathbf{Y}_1(t) = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \mathbf{Y}_1(t) = e^{-2t} \begin{pmatrix} -2\cos 3t - 3\sin 3t \\ 3\cos 3t - 2\sin 3t \end{pmatrix}.$$

Since the two sides of the differential equation agree, the function $\mathbf{Y}_1(t)$ is a solution.

Using $\mathbf{Y}_2(t)$, we have

$$\begin{aligned} \frac{d\mathbf{Y}_2}{dt} &= -2e^{-2t} \begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix} + e^{-2t} \begin{pmatrix} -3\cos 3t \\ -3\sin 3t \end{pmatrix} \\ &= e^{-2t} \begin{pmatrix} 2\sin 3t - 3\cos 3t \\ -3\sin 3t - 2\cos 3t \end{pmatrix}, \end{aligned}$$

and

$$\mathbf{A}\mathbf{Y}_2(t) = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \mathbf{Y}_2(t) = e^{-2t} \begin{pmatrix} 2\sin 3t - 3\cos 3t \\ -3\sin 3t - 2\cos 3t \end{pmatrix}.$$

The two sides of the differential equation agree. Hence, $\mathbf{Y}_2(t)$ is also a solution.

Since both $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions, we proceed to part (b).

(b) Note that $\mathbf{Y}_1(0) = (1, 0)$ and $\mathbf{Y}_2(0) = (0, 1)$, and these vectors are not on the same line through the origin. Hence, $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are linearly independent, and we proceed to part (c) of the exercise.

(c) To find the solution with the initial condition $\mathbf{Y}(0) = (2, 3)$, we must find constants k_1 and k_2 so that

$$k_1\mathbf{Y}_1(0) + k_2\mathbf{Y}_2(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

We have $k_1 = 2$ and $k_2 = 3$, and the solution with initial condition (2, 3) is

$$\mathbf{Y}(t) = e^{-2t} \begin{pmatrix} 2\cos 3t - 3\sin 3t \\ 2\sin 3t + 3\cos 3t \end{pmatrix}.$$

29. **(a)** First, we check to see if $\mathbf{Y}_1(t)$ is a solution. The left-hand side of the differential equation is

$$\frac{d\mathbf{Y}_1}{dt} = \begin{pmatrix} e^{-t} + 36e^{3t} \\ -e^{-t} + 12e^{3t} \end{pmatrix},$$

and the right-hand side is

$$\mathbf{A}\mathbf{Y}_1(t) = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -e^{-t} + 12e^{3t} \\ e^{-t} + 4e^{3t} \end{pmatrix} = \begin{pmatrix} e^{-t}36e^{3t} \\ -e^{-t} + 12e^{3t} \end{pmatrix}.$$

Consequently, $\mathbf{Y}_1(t)$ is a solution. However,

$$\frac{d\mathbf{Y}_2}{dt} = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix},$$

and

$$\mathbf{A}\mathbf{Y}_2(t) = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -e^{-t} \\ 2e^{-t} \end{pmatrix} = \begin{pmatrix} 4e^{-t} \\ -e^{-t} \end{pmatrix}.$$

Consequently, the function $\mathbf{Y}_2(t)$ is not a solution. In this case, we are not able to solve the given initial-value problem, so we stop here.

30. **(a)** This holds in all dimensions. In two dimensions the computation is

$$\mathbf{A}k\mathbf{Y} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} k \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} kx \\ ky \end{pmatrix}$$

$$= \begin{pmatrix} akx + bky \\ ckx + dky \end{pmatrix}$$

$$= k \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = k\mathbf{AY}.$$

(b) To verify the first half of the Linearity Principle, we suppose that $\mathbf{Y}_1(t) = (x_1(t), y_1(t))$ is a solution to the system

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

and that k is an any constant. In order to verify that the function $\mathbf{Y}_2(t) = k\mathbf{Y}_1(t)$ is also a solution, we need to substitute $\mathbf{Y}_2(t)$ into both sides of the differential equation and check for equality. In other words, after we write $\mathbf{Y}_2(t)$ in scalar notation as $\mathbf{Y}_2(t) = (x_2(t), y_2(t))$, we must show that

$$\frac{dx_2}{dt} = ax_2 + by_2$$
$$\frac{dy_2}{dt} = cx_2 + dy_2$$

given that we know that

$$\frac{dx_1}{dt} = ax_1 + by_1$$
$$\frac{dy_1}{dt} = cx_1 + dy_1.$$

Since $x_2(t) = kx_1(t)$ and $y_2(t) = ky_1(t)$, we can multiply both sides of

$$\frac{dx_1}{dt} = ax_1 + by_1$$
$$\frac{dy_1}{dt} = cx_1 + dy_1.$$

by k to obtain

$$k\,\frac{dx_1}{dt} = k(ax_1 + by_1)$$
$$k\,\frac{dy_1}{dt} = k(cx_1 + dy_1).$$

However, using standard algebraic properties and the rules of differentiation, this system is equivalent to

$$\frac{d(k\,x_1)}{dt} = a(kx_1) + b(ky_1)$$
$$\frac{d(k\,y_1)}{dt} = c(kx_1) + d(ky_1),$$

which is the same as the desired equality

$$\frac{dx_2}{dt} = ax_2 + by_2$$
$$\frac{dy_2}{dt} = cx_2 + dy_2.$$

To verify the second half of the Linearity Principle, we suppose that $\mathbf{Y}_1(t) = (x_1(t), y_1(t))$ and $\mathbf{Y}_2(t) = (x_2(t), y_2(t))$ are solutions to the system

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy.$$

To verify that the function $\mathbf{Y}_3(t) = \mathbf{Y}_1(t) + \mathbf{Y}_2(t)$ is also a solution, we need to substitute $\mathbf{Y}_3(t)$ into both sides of the differential equation and check for equality. In other words, after we write $\mathbf{Y}_3(t)$ in scalar notation as $\mathbf{Y}_3(t) = (x_3(t), y_3(t))$, we must show that

$$\frac{dx_3}{dt} = ax_3 + by_3$$
$$\frac{dy_3}{dt} = cx_3 + dy_3$$

given that we know that

$$\frac{dx_1}{dt} = ax_1 + by_1$$
$$\frac{dy_1}{dt} = cx_1 + dy_1$$

and

$$\frac{dx_2}{dt} = ax_2 + by_2$$
$$\frac{dy_2}{dt} = cx_2 + dy_2.$$

Adding the two given systems together yields the system

$$\frac{dx_1}{dt} + \frac{dx_2}{dt} = ax_1 + by_1 + ax_2 + by_2$$
$$\frac{dy_1}{dt} + \frac{dy_2}{dt} = cx_1 + dy_1 + cx_2 + dy_2,$$

which can be rewritten as

$$\frac{d(x_1 + x_2)}{dt} = a(x_1 + x_2) + b(y_1 + y_2)$$
$$\frac{d(y_1 + y_2)}{dt} = c(x_1 + x_2) + d(y_1 + y_2).$$

But this last system of equalities is the desired equality that indicates that $\mathbf{Y}_3(t)$ is also a solution.

31. **(a)** If $(x_1, y_1) = (0, 0)$, then (x_1, y_1) and (x_2, y_2) are on the same line through the origin because (x_1, y_1) is the origin. So (x_1, y_1) and (x_2, y_2) are linearly dependent.

(b) If $(x_1, y_1) = \lambda(x_2, y_2)$ for some λ, then (x_1, y_1) and (x_2, y_2) are on the same line through the origin. To see why, suppose that $x_2 \neq 0$ and $\lambda \neq 0$. (The $\lambda = 0$ case was handled in part (a) above.) In this case, $x_1 \neq 0$ as well. Then the slope of the line through the origin and (x_1, y_1) is y_1/x_1, and the slope of the line through the origin and (x_2, y_2) is y_2/x_2. However, because $(x_1, y_1) = \lambda(x_2, y_2)$, we have

$$\frac{y_1}{x_1} = \frac{\lambda y_2}{\lambda x_2} = \frac{y_2}{x_2}.$$

Since these two lines have the same slope and both contain the origin, they are the same line. (The special case where $x_2 = 0$ reduces to considering vertical lines through the origin.)

(c) If $x_1 y_2 - x_2 y_1 = 0$, then $x_1 y_2 = x_2 y_1$. Once again, this condition implies that (x_1, y_1) and (x_2, y_2) are on the same line through the origin. For example, suppose that $x_1 \neq 0$, then

$$y_2 = \frac{x_2 y_1}{x_1} = \frac{x_2}{x_1} y_1.$$

But we already know that

$$x_2 = \frac{x_2}{x_1} x_1,$$

so we have

$$(x_2, y_2) = \frac{x_2}{x_1}(x_1, y_1).$$

By part (b) above (where $\lambda = x_2/x_1$), the two vectors are linearly dependent.

If $x_1 = 0$ but $y_1 \neq 0$, it follows that $x_2 y_1 = 0$, and thus $x_2 = 0$. Thus, both (x_1, y_1) and (x_2, y_2) are on the vertical line through the origin.

Finally, if $x_1 = 0$ and $y_1 = 0$, we can use part (a) to show that the two vectors are linearly dependent.

32. If $x_1 y_2 - x_2 y_1$ is nonzero, then $x_1 y_2 \neq x_2 y_1$. If both $x_1 \neq 0$ and $x_2 \neq 0$, we can divide both sides by $x_1 x_2$, and we obtain

$$\frac{y_2}{x_2} \neq \frac{y_1}{x_1},$$

and therefore, the slope of the line through the origin and (x_2, y_2) is not the same as the slope of the line through the origin and (x_1, y_1).

If $x_1 = 0$, then $x_2 \neq 0$. In this case, the line through the origin and (x_1, y_1) is vertical, and the line through the origin and (x_2, y_2) is not vertical.

33. The initial position of $\mathbf{Y}_1(t)$ is $\mathbf{Y}_1(0) = (-1, 1)$. By the Linearity Principle, we know that $k\mathbf{Y}_1(t)$ is also a solution of the system for any constant k. Hence, for any initial condition of the form $(-k, k)$, the solution is $k\mathbf{Y}_1(t)$.

(a) The curve $2\mathbf{Y}_1(t) = (-2e^{-t}, 2e^{-t})$ is the solution with this initial condition.

(b) We cannot find the solution for this initial condition using only $\mathbf{Y}_1(t)$.

(c) The constant function $0\mathbf{Y}_1(t) = (0, 0)$ (represented by the equilibrium point at the origin) is the solution with this initial condition.

(d) The curve $-3\mathbf{Y}_1(t) = (3e^{-t}, -3e^{-t})$ is the solution with this initial condition.

34. **(a)** If $\mathbf{Y}(t) = (t, t^2/2)$, then $x(t) = t$ and $y(t) = t^2/2$. Then $dx/dt = 1$, and $dy/dt = t = x$. So $\mathbf{Y}(t)$ satisfies the differential equation.

(b) For $2\mathbf{Y}(t)$, we have $x(t) = 2t$, and $y(t) = t^2$. In this case, we need only consider $dx/dt = 2$ to see that the function is not a solution to the system.

35. **(a)** Using the Product Rule we compute

$$\frac{dW}{dt} = \frac{dx_1}{dt}y_2 + x_1\frac{dy_2}{dt} - \frac{dx_2}{dt}y_1 - x_2\frac{dy_1}{dt}.$$

(b) Since $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are solutions, we know that

$$\frac{dx_1}{dt} = ax_1 + by_1$$
$$\frac{dy_1}{dt} = cx_1 + dy_1$$

and that

$$\frac{dx_2}{dt} = ax_2 + by_2$$
$$\frac{dy_2}{dt} = cx_2 + dy_2.$$

Substituting these equations into the expression for dW/dt, we obtain

$$\frac{dW}{dt} = (ax_1 + by_1)y_2 + x_1(cx_2 + dy_2) - (ax_2 + by_2)y_1 - x_2(cx_1 + dy_1).$$

After we collect terms, we have

$$\frac{dW}{dt} = (a + d)W.$$

(c) This equation is a homogeneous, linear, first-order equation (as such it is also separable—see Sections 1.1, 1.2, and 1.8). Therefore, we know that the general solution is

$$W(t) = Ce^{(a+d)t}$$

where C is any constant (but note that $C = W(0)$).

(d) From Exercises 31 and 32, we know that $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are linearly independent if and only if $W(t) \neq 0$. But, $W(t) = Ce^{(a+d)t}$, so $W(t) = 0$ if and only if $C = W(0) = 0$. Hence, $W(t) = 0$ is zero for some t if and only if $C = W(0) = 0$.

EXERCISES FOR SECTION 3.2

1. **(a)** The characteristic polynomial is

$$(3 - \lambda)(-2 - \lambda) = 0,$$

and therefore the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 3$.