MAT 2 -110 Great Mathematical Ideas

Exam 1 October 12, 2016

Solution

NAME

This exam is closed book, closed notes, calculators are allowed but no other devices. 80 pts. possible.

1. (~~2 pts.~~ each) True or false? Circle one.

True False a. Dynamical systems given by linear functions can exhibit chaotic behavior.

True False b. The Newtonian universe is one of cause and effect.

True False c. Modern computers have capabilities equal to Laplace's Demon.

True False d. A function can only have one output for each input.

2. (2 pts. each) Fill in the blanks.

a. A function whose rule involves an element of chance is called stochastic.

b. Another word for initial condition of an iterate is a(n) seed. Another word for orbit of an initial condition is a(n) itinerary.

c. The phenomenon of sensitive dependence on initial conditions (SDIC) is more colloquially known as the butterfly effect.

d. A stable fixed point is also known as a(n) attractor.

3. (5 pts.) a. Let f(x) = 0.5 x + 4. What are the first 3 iterates of the seed $x_0 = 4$?

$x_0 = 4$

$x_1 = .5 \times 4 + 4 = 2 + 4 = 6$

$x_2 = .5 \times 6 + 4 = 3 + 4 = 7$

$x_3 =$

b. Let $g(x) = x^2 - 2$. What are the first 3 iterates of the seed $x_0 = 2$?

$x_0 = 2$

$x_1 = 2^2 - 2 = 2$

$x_2 = x_3 = 2$

4. (8 pts.) a. Given the function g(x) = 2 x (1 – x), use algebra to find a fixed point of g(x). Show your work.

$$g(x^*) = x^*$$

$$2x^*(1-x^*) = x^*$$

$$2x^* - 2x^{*2} = x^*$$

subt. x^*: $$x^* - 2x^{*2} = 0$$

$$x^*(1-2x^*) = 0$$

$$x^* = 0 \quad \text{or} \quad (1-2x^*) = 0$$

$$2x^* = 1$$

$$x^* = 1/2$$

b. What kind of physical system does this function model?

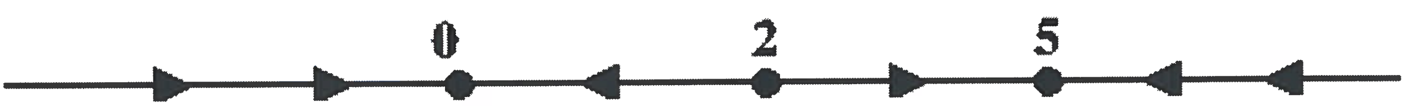
Logistic population growth.

Population growth where there is a carrying capacity.

5. (3 pts. each)

a. The dynamics of a function is described by the phase line shown above. There are three fixed points. Describe the fixed points' stability.

0 stable 2 unstable 5 stable



b. For an iterated function described by the above phase line, what is the long-term behavior of the seed 3?

orbit goes to 5.

6. (4 pts.) a. Give the 4 condition for a dynamic system to be chaotic:

i. Deterministic

ii. Aperiodic orbits

iii. Bounded.

iv. SDIC

b. (4 pts.) Analyze each of the four factors above for the iterative function system given by $f(x) = 2x$. Is it chaotic?

i. It is deterministic, this rule doesn't involve chance
ii. It has ~~a~~ aperiodic orbits, the values don't repeat
iii) It is not bounded. For example $1 \to 2 \to 4 \to 8 \to 16 \ldots \to \infty$
iv) It has SDIC. 2 close values 0 & .1 split $0 \to 0$, $.1 \to \infty$

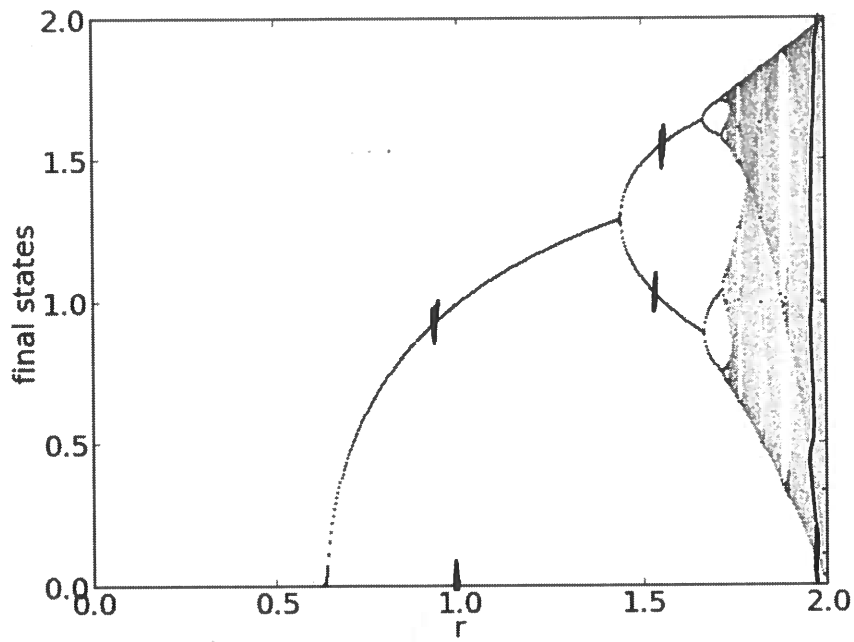
It is not chaotic, fails #3.

c. (4 pts.) In class, I made pink colored playdoh by introducing a few drops of red food coloring in one spot and proceeded as you remember. Analyze each of the four factors of chaos above for this physical system. Is it chaotic?

i. It is deterministic: stretch, fold, repeat.
ii) It is aperiodic, every spot in the playdoh gets color
iii) It is bounded because of the fold.
iv) The color starts close & ends up far so SDIC

It is chaotic.

7. (6 pts.) The bifurcation diagram for an iterated function (not the logistic equation) is shown in the figure below:



a. Circle the statement that best describes the long-term behavior of orbits of this dynamical system if r=1.0?
- The orbit approaches zero
- The orbit approaches a fixed point near x = 1.
- The orbit approaches a fixed point near x = 1.3.
- The orbit is periodic with period 2
- The orbit appears to be aperiodic.

b. Circle the statement that best describes the long-term behavior of orbits of this dynamical system if r=1.5?
- The orbit approaches zero
- The orbit approaches a fixed point near x= 1.
- The orbit approaches a fixed point near x = 1.3.
- The orbit is periodic with period two.
- The orbit appears to be aperiodic.

c. Circle the statement that best describes the long-term behavior of orbits of this dynamical system if r=1.9?
- The orbit approaches zero.
- The orbit approaches a fixed point at x = 1.
- The orbit approaches a fixed point near x = 1.3.
- The orbit is periodic with period two.
- The orbit appears to be aperiodic.

8. (3 pts.) In the table below are the first 7 iterates of a function with two different initial conditions. Do you think this function exhibits SDIC? Justify your answer.

t	xt	yt	$X_t - Y_t$
0	0.60	0.61	-.01
1	0.91	0.90	.01
2	0.30	0.33	-.03
3	0.81	0.84	-.03
4	0.59	0.51	.08
5	0.29	0.18	.09
6	0.79	0.56	.23

yes. Points that start close (dist .01) move apart (dist .23)

9. (7 pts.) Suppose we analyze a dynamical system that exhibits a period doubling route to chaos. This dynamical system undergoes a bifurcation from period one to period two at r=7. The system undergoes a bifurcation from period two to period four at r=9, and there is a bifurcation from period four to period eight at r=9.43.

a. Calculate Δ_1, Δ_2, and δ_1 as we did in class to estimate the Feigenbaum constant.

$$\Delta_1 = 9-7=2$$
$$\Delta_2 = 9.43-9=.43$$
$$\delta_1 \doteq \frac{\Delta_1}{\Delta_2} \doteq 4.651$$

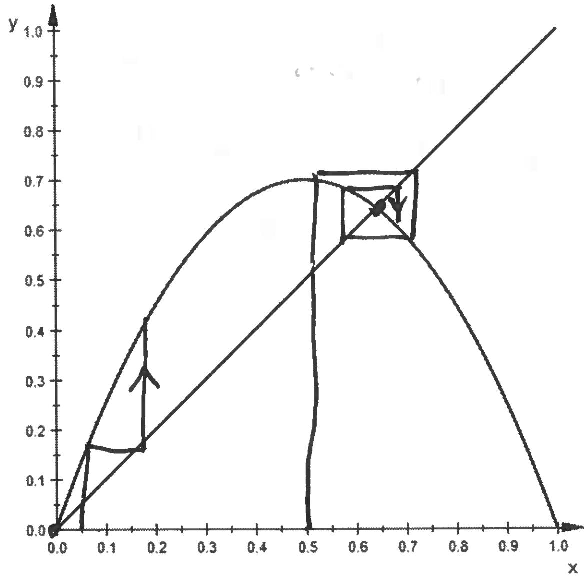
b. Why is the Feigenbaum constant considered "universal" in the way that π is universal?

All period doubling to chaos dyn. systems have it.

c. Name a physical system that exhibits a period doubling route to chaos.

Double pendulum; dripping faucet

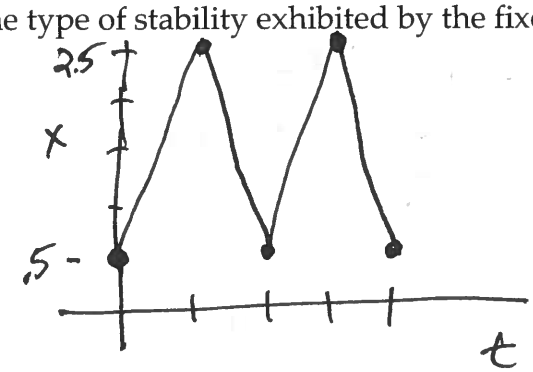
10. (6 pts.) Identify and classify the fixed point(s) of the function given by the curved line below:

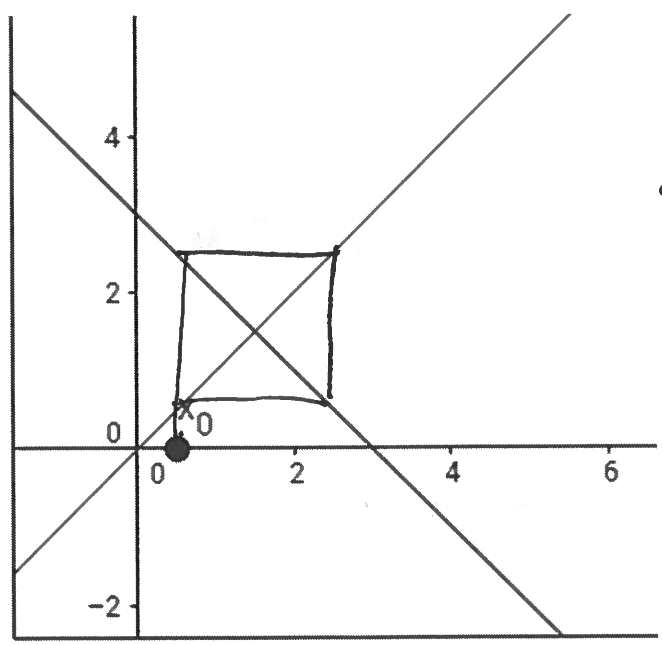


0 unstable.

~.65 stable

11. (8 pts.) Below is the graph of the function f(x) = -x + 3 together with the line y = x. Use graphical iteration to produce 4 iterates with initial condition x0 = .5. To the right, make a time series graph of your iterates. Describe the type of stability exhibited by the fixed point for this function.





3/2 is neutral.