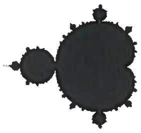
MAT 2 -110 Great Mathematical Ideas

Exam 2 October 21, 2016



Solution

NAME

This exam is closed book, closed notes, calculators are allowed but no other devices. 80 pts. possible.

1. (2 pts. each) True or false? Circle one.

True False a. The set { …,-3,-2,-1,0,1,2,3, …} is countably infinite.

True False b. The set of all rational numbers is uncountably infinite.

True False c. The Cantor set is uncountably infinite.

True False d. The set of all real numbers x where x > 2.1 and x < 2.100001 is uncountably infinite.

True False e. The point -0.6 + 0.2 i is in the Mandelbrot set.

2. (2 pts. each) Fill in the blanks.

a. A transformation that has the effect of stretching, shrinking, shearing, rotating, or moving a geometric object is called a(n) affine transformation.

b. The Cantor set consists of all numbers between 0 and 1 whose base-3 (ternary) expansion does not have any 1's.

c. A Julia set that is not one single connected shapes is called a(n) dust.

d. By definition, i is the imaginary number that satisfies $i^2 =$ -1.

3. (6 pts.) The function $f(z) = z^2 + c$, where z and c are complex numbers, is used in defining the Mandlebrot set (MS). We have seen two different but equivalent definitions of what is required for a given c to be in the MS. Give these criteria that define the MS :

a. The Julia set associated with c is connected.

b. The iterates starting at 0+0i converge/are bounded.

DNBU.

4. (3 pts) Matching. Match the fractal to its self-similarity dimension.
Choices: I. 0.6309 II. 1.2619 III. 2.7268

__III__ a. Menger sponge

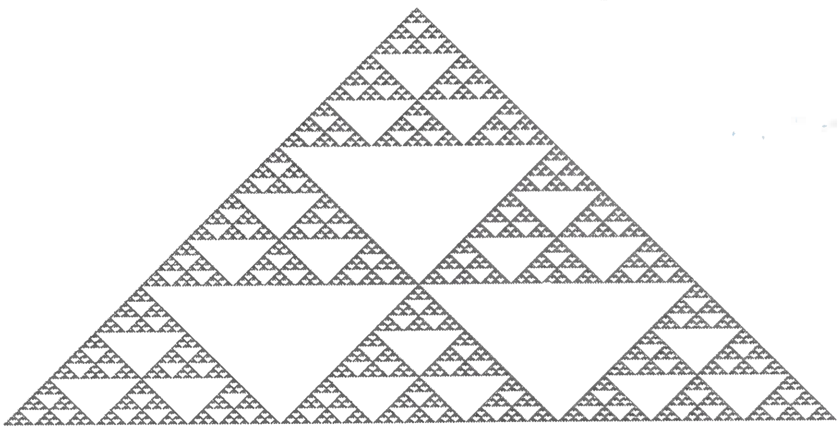
__I__ b. Cantor set

__II__ c. Koch curve

5. (2 pts.) What is the first sentence of Cantor's proof that the interval (0,1) is uncountably infinite.

Suppose (0,1) is countably infinite

6. (9 pts.) a. What is the self-similarity dimension of the fractal shown below? Note that this is similar, but not identical to, the Sierpiński triangle. Show your work!



"Pascal triangle modulo 3" by Alexis Monnerot-Dumaine

$6 = 3^D$

$D = \frac{\log 6}{\log 3}$

$\doteq 1.6309$

b. Suppose that the original triangle has area 1. What is an expression for the area of the first iterate (the "generator")? What is an expression for the area of the nth iterate? What can you conclude about the area of the resultant fractal?

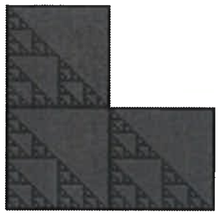
$\frac{2}{3}$

Area goes to 0.

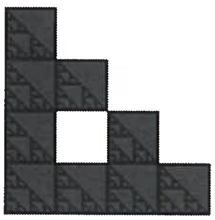
$\left(\frac{2}{3}\right)^n$

7. (12 pts.)This problem concerns box counting dimension.

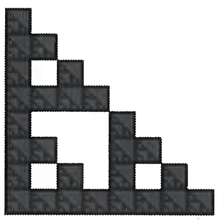
a. Here we cover a right Sierpinski gasket with smaller and smaller boxes. (The r's are the size of the boxes.) On the graph at the right, produce the log-log graph for this data. Include axis labels.



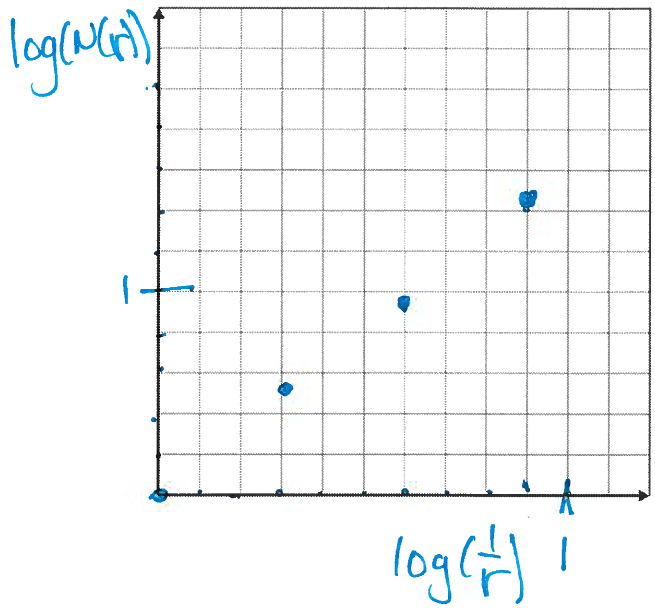
$r_1 = 1/2, N(r_1) = 3$



$r_2 = 1/4, N(r_2) = 9$



$r_3 = 1/8, N(r_3) = 27$



$\log(\frac{1}{r})$	log N
0	0
.301	.477
.602	.95
.903	1.43

b. If you make a log-log plot for a box count and the points fail to fall along a straight line, what is the likely cause of the problem and its cure?

Boxes aren't small enough.

c. Self-similarity dimension and box-counting dimension give the same value for many fractal objects. Why do we bother with the more complicated box-counting dimension?

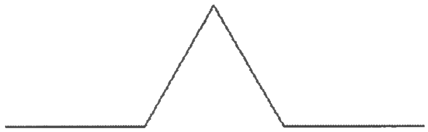
Box counting works on natural fractals.
random fractals
etc.

d. Suppose it takes 100 boxes of side 1/2 and 300 boxes of side 1/4 to cover a certain object. What is the box-counting dimension of this object? Show your work

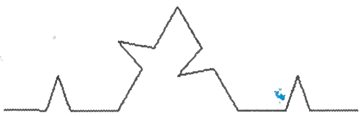
.301	2
.602	2.477

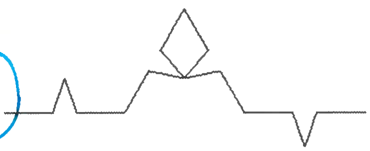
$$\frac{\Delta y}{\Delta x} = \frac{.477}{.301} = 1.585$$

8. (3 pts.) Suppose that after the first step in generating a random Koch curve one has a shape that looks like this:



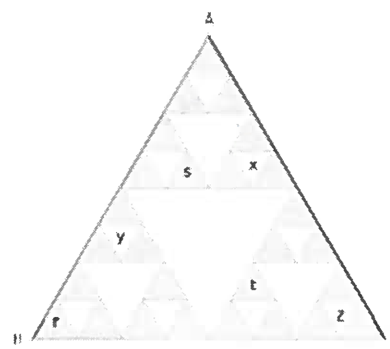
i.e., the first coin toss was Heads, so the first bend is up. If the next four coin tosses are Heads, Tails, Tails, Tails, what would the shape look like at step n=2? Circle the correct answer

A 

B 

C 

9. (3 pts.) Suppose one plays Barnsley's chaos game and the starting point is exactly on the top corner, labeled A in the figure to the right. What sequence of moves leads to the point landing inside of (and not on the boundary of) the triangle marked r?



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10. (6 pts.) a. Evaluate $(2 + 3i)^2$ Show your work

$4 - 9 + 2 \cdot 2 \cdot 3i = -5 + 12i$

b. Simplify i^{41} $i^2 = -1$ $i^4 = 1$ $i^{41} = i^{40} \cdot i = 1 \cdot i = i$

c. Suppose z is a complex number with r = 2 and θ = 45°. Find z^2 in r, θ form.

$r = 4$

$\theta = 90^\circ$

11. (9 pts.) Consider the function $f(x) = x^2 - 1$, where x is a real number. Calculate 3 iterates for the following seeds and determine if these points are in the (real) Julia set of this function.

a. $x_0 = -2$

3, 8, 63 not in J.S

b. $x_0 = 0$

-1, 0, -1 in J.S

c. $x_0 = .5$

-.75, -.4375, -.8086 in J.S.

12. (8 pts.)

a. $1 + 0i$ is not in the Mandelbrot set. Carefully explain why.

Iterates of 0+0i under z^2+1

0, 1, 2, 5, 26, ...

Diverges so not in M.S

b. $-1 + 0i$ is in the Mandelbrot set. Carefully explain why.

Iter. of 0 under z^2-1

0, -1, 0, -1, ...

Doesn't div. so in M.S.